# Relations

Murat Osmanoglu

# Relations

- For a cartesian product set $A \times B = \{(x, y) | x \in A \ \wedge \ y \in B\}$, a binary relation from A to B is a subset of $A \times B$, i.e. $R \subseteq A \times B$

- if $(\mathrm{a}, \mathrm{b}) \in R$, then a is said to be related to b by R, i.e $\mathrm{a}Rb$

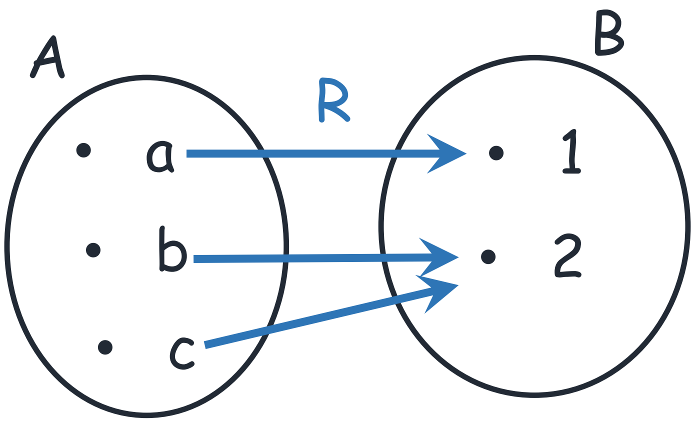
- Let A be the set of students and B be the set of courses

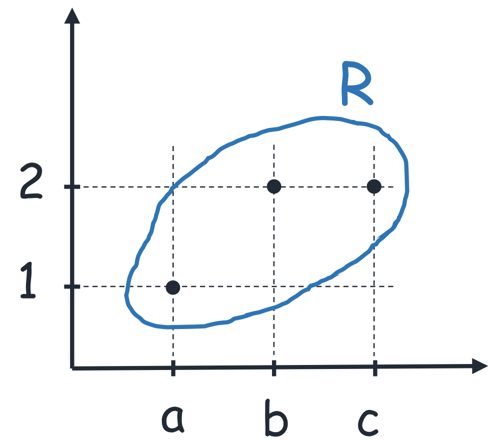
  A = {Ahmet, Efe, Buse, Pelin, . . .}
  B = {Math, Physics, Discrete, Algorithms, . . .}

  Let R be the relation such that if student a is taking course b, $(\mathrm{a}, \mathrm{b}) \in R$.

  (Ahmet, Physics) $\in R$, (Efe, Discrete) $\notin R$

# Relations




| R | 1 | 2 |
|---|---|---|
| a | 1 | 0 |
| b | 0 | 1 |
| c | 0 | 1 |

R = {(a, 1), (b, 2), (c, 2)}

- the number of relations that can be defined from A to B :

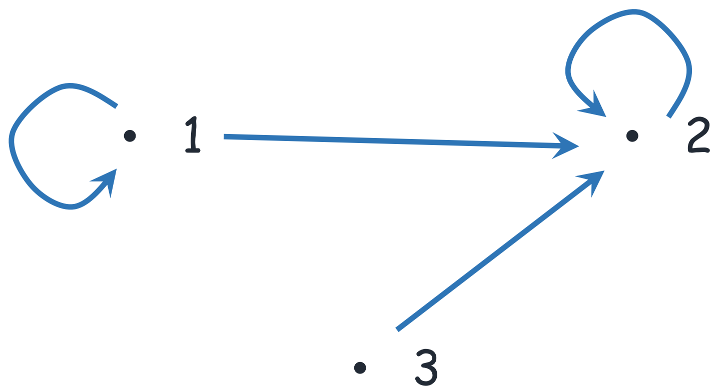
$$2^{|A||B|}$$

# Relations

- A relation can be defined on a single set $A$ as a subset of $A \times A$

$A = \{1, 2, 3\}$

$R = \{(1, 1), (1, 2), (2, 2), (3, 2)\}$



# Functions as Relations

$$R \subseteq A \times B$$

domain (A) — range (B)

R(A) : the image of R, $R(A) = \{y \in B | (x, y) \in R, \exists x \in A\}$

Function is a relation that satisfies two conditions :

- for every element x of the domain, there is an element y in the range such that (x,y) is an element of the relation

  Let $R \subseteq A \times B$ be the relation, $\forall x[(x \in A) \rightarrow (\exists y \in B \;\; s.t. (x, y) \in R)]$

- for every element x of the domain, there is only one element y of the range such that (x,y) is an element of the relation

  Let $R \subseteq A \times B$ be the relation, $\forall x[((x, y_1) \in R \wedge (x, y_2) \in R) \rightarrow (y_1 = y_2)]$

# Properties

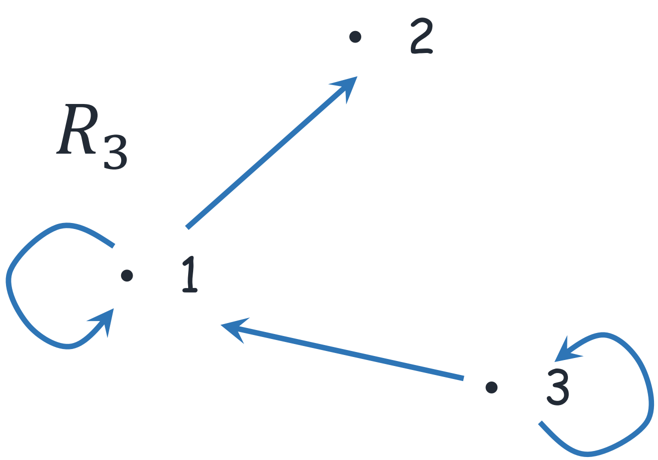
## Reflexivity

- A relation on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$

$$R_1 = \{(1, 1), (1, 2), (2, 2), (3, 2), (3, 3)\}$$

| $R_2$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 0 | 1 | 0 |
| 3 | 0 | 1 | 1 |

$R_3$



# Properties

## Symmetry

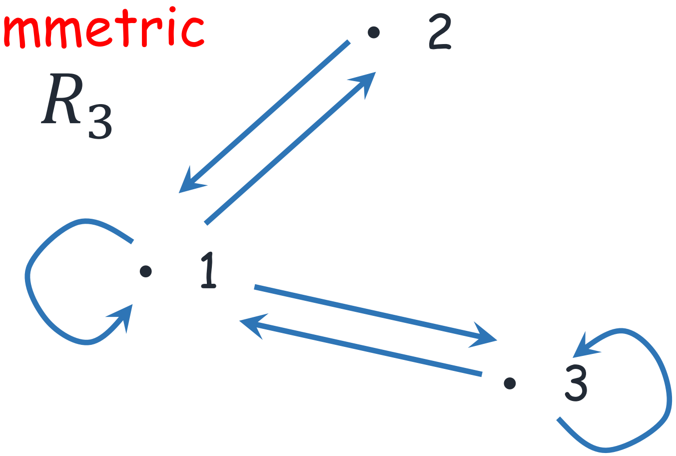
- A relation on a set A is called symmetric if $(a, b) \in R$, then $(b, a) \in R$
- If the relation is not symmetric, it is called asymmetric.
- If for all $(a, b) \in R$, $(b, a) \notin R$ or $a = b$, then it is called antisymmetric

antisymmetric

| $R_2$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 0 | 1 | 1 |
| 3 | 1 | 0 | 1 |

symmetric

$R_3$




$R_1$ = {(1, 1), (1, 2), (2, 1), (3, 2), (3, 3)}

asymmetric

# Properties

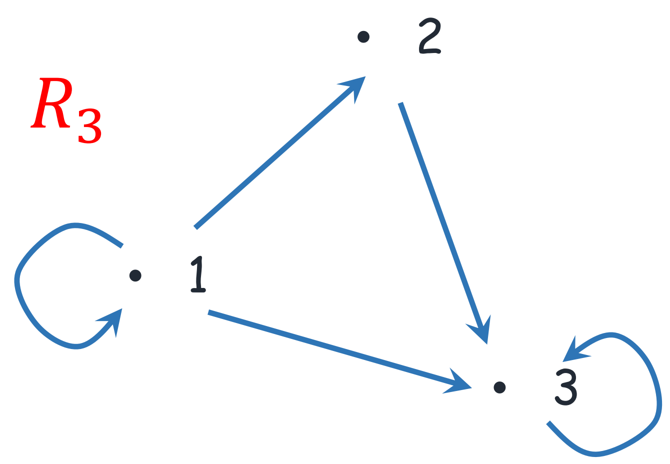
## Transitivity

- A relation on a set $A$ is called symmetric if $(a, b) \in R \wedge (b, c) \in R$, then $(a, c) \in R$

$R_1$ = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 1)}

| $R_2$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 0 | 1 | 1 |
| 3 | 1 | 0 | 1 |

$R_3$



# Properties

Let R be a relation on Z such that $(a, b) \in R$ if $a.b \geq 0$

- Since $a . a \geq 0$ for all $a \in Z$, $(a, a) \in R$ for all $a \in Z$. Thus, $R$ is reflexive.

- $[(a, b) \in R] \rightarrow (a.b \geq 0) \rightarrow (b.a \geq 0)$

  $\rightarrow R$ is symmetric

- $[(a, b) \in R \land (b, c) \in R] \rightarrow [(a.b \geq 0) \land (b.c \geq 0)]$

  $\rightarrow (a.b.b.c \geq 0)$

  $\rightarrow (a.c \geq 0)$

  $\rightarrow (a, c) \in R$

  $\rightarrow R$ is transitive

# Properties

Consider the division operator, $'|'$, as a relation on integers :

$$(a, b) \in '|' \rightarrow a \mid b$$

- Since $a \mid a$, $(a, a) \in '|'$ for all $a \in Z$. Thus, $'|'$ is reflexive.
- $[(a, b) \in '|'] \rightarrow (a \mid b) \rightarrow$ (either $a = b$ or $\mathrm{b} \nmid a$)

  $\rightarrow '|'$ is antisymmetric

- $[(a, b) \in '|' \wedge (b, c) \in '|'] \rightarrow [(a \mid b) \wedge (b \mid c)]$

  $\rightarrow [b = x.a \wedge \mathrm{c} = y.b, \ \exists x, y \in Z]$

  $\rightarrow (\mathrm{c} = x.y.a)$

  $\rightarrow a \mid c \rightarrow (a, c) \in '|'$

  $\rightarrow '|'$ is transitive

# Properties

How many reflexive relations can be defined on a set A of n elements?

- A = {1, 2, . . ., n}
- there are |AxA| = $n^2$ pairs
- a reflexive relation must contain the pairs (1, 1), . . ., (n, n)
- take these pairs out, $(n^2 - n)$ remaining pairs
- $2^{(n^2-n)}$ different relations can be formed with the $(n^2 - n)$ remaining pairs
- add each of them the pairs (1, 1), . . ., (n, n) to make them reflexive

# Properties

How many symmetric relations can be defined on a set A of n elements?

- A = {1, 2, . . ., n}
- there are |AxA| = $n^2$ pairs

$A_1 = \{(a_i, a_i) | 1 \leq i \leq n\}$

$|A_1| = n$

$A_2 = \{(a_i, a_j) | 1 \leq i, j \leq n \text{ and } i \neq j\}$

$|A_2| = n^2 - n$

$A_3 = \{(a_i, a_j) | 1 \leq i, j \leq n, i \neq j, \text{and } (a_j, a_i) \in A_3\}$

$|A_3| = (n^2 - n)/2$

$\{\ldots, \ldots\}$

$$\left(2^n . 2^{\frac{n^2-n}{2}}\right) = 2^{(n^2+n)/2}$$

# Operations

Union : Given $R, S \subseteq A \times B$,

$$T = R \cup S = \{(x, y) | (x, y) \in R \ \lor \ (x, y) \in S\}$$

Intersection : Given $R, S \subseteq A \times B$,

$$T = R \cap S = \{(x, y) | (x, y) \in R \ \land \ (x, y) \in S\}$$

Complement : Given $R \subseteq A \times B$,

$$T = \bar{R} = \{(x, y) | (x, y) \notin R \ \}$$

Inverse : Given $R \subseteq A \times B$,

$$T = R^{-1} = \{(y, x) \in B \times A | (x, y) \in R \ \}$$

Composition : Given $R \subseteq A \times B$ and $S \subseteq B \times C$

$$T = S \circ R = \{(x, z) | (x, y) \in R \ \land \ (y, z) \in S\}$$

# Operations

| R | 1 | 2 |
|---|---|---|
| a | 1 | 0 |
| b | 0 | 1 |
| c | 1 | 0 |

| S∘R | u | v |
|---|---|---|
| a | 0 | 0 |
| b | 1 | 1 |
| c | 0 | 0 |

| S | u | v |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 1 | 1 |

| $S^{-1}$ | 1 | 2 |
|---|---|---|
| u | 0 | 1 |
| v | 0 | 1 |

# Operations

- $A = \{1, 2, 3\}$, R = { (1, 1), (2, 1), (3, 2) }

| R | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 |
| 3 | 0 | 1 | 0 |

| R | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 |
| 3 | 0 | 1 | 0 |

| R∘R | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 |
| 3 | 0 | 1 | 0 |

- $R^2 = R \circ R = \{(1,1), (2,1), (3,2)\}$
  $R^3 = R^2 \circ R = \{(1,1), (2,1), (3,2)\}$

- The relation R on a set $A$ is transitive if and only if $R^n \subseteq R$ for some $n \in Z^+$

# Equivalence Relations

Definition : A relation R on a set A is called an equivalence relation if it's reflexive, symmetric, and transitive. If $(a, b) \in R$, then a and b are called equivalent, i.e. a~b.

- Let R be a relation defined on real numbers such that $(a, b) \in R$ if and only if a - b is an integer. R is an equivalence relation ?
  - $\forall a \in \mathbb{R}$, since $a - a = 0 \in \mathbb{Z}$, $(a, a) \in R$ (reflexive)
  - $[(a, b) \in R] \rightarrow [a - b \in \mathbb{Z}]$
    $\rightarrow [b - a \in \mathbb{Z}] \rightarrow [(b, a) \in R]$ (symmetric)
  - $[(a, b) \in R \wedge (b, c) \in R] \rightarrow [a - b \in \mathbb{Z} \wedge b - c \in \mathbb{Z}]$
    $\rightarrow [a - c \in \mathbb{Z}] \rightarrow [(a, c) \in R]$ (transitive)

# Equivalence Relations

Definition : A relation R on a set A is called an equivalence relation if it's reflexive, symmetric, and transitive. If $(a, b) \in R$, then a and b are called equivalent, i.e. a~b.

- Let R be a relation defined on integers such that $(a, b) \in R$ if and only if $a \equiv b (mod\ m)$. R is an equivalence relation ?
  - $\forall a \in \mathbb{Z}$, since $a \equiv a (mod\ m)$, $(a, a) \in R$ (reflexive)
  - $[(a, b) \in R] \rightarrow [a \equiv b (mod\ m)]$
    $\rightarrow [b \equiv a (mod\ m)] \rightarrow [(b, a) \in R]$ (symmetric)
  - $[(a, b) \in R \wedge (b, c) \in R] \rightarrow [a \equiv b (mod\ m) \wedge b \equiv c (mod\ m).]$
    $\rightarrow [a \equiv c (mod\ m)] \rightarrow [(a, c) \in R]$
    (transitive)

# Equivalence Relations

Definition : A relation R on a set A is called an equivalence relation if it's reflexive, symmetric, and transitive. If $(a, b) \in R$, then a and b are called equivalent, i.e. a~b.

- Let R be a relation defined on real numbers such that $(a, b) \in R$ if and only if $|a - b| < 1$. R is an equivalence relation ?

  - $\forall a \in \mathbb{Z}$, since $|a - a| = 0 < 1, (a, a) \in R$ (reflexive)

  - $[(a, b) \in R] \rightarrow [|a - b| < 1]$
    $\rightarrow [|b - a| < 1] \rightarrow [(b, a) \in R]$ (symmetric)

  - $[(a, b) \in R \wedge (b, c) \in R] \rightarrow [|a - b| < 1 \wedge |b - c| < 1]$

    for $a = 1, b = \frac{1}{10}$, and $c = -\frac{2}{10}$
    $|a - b| < 1$ and $|b - c| < 1$, but $|a - c| > 1$ (not transitive)

# Equivalence Relations

Definition : Let R be an equivalence relation on a set A. The set of all elements related to an element a is called the equivalence class of a, denoted by $[a]_R$

$$[a]_R = \{s \in A | (a, s) \in R\}$$

- What are the equivalence classes of 2 and 1 for the congruence relation of module 5 ?
  - $(2, s) \in R \rightarrow 2 \equiv s(mod\, 5) \rightarrow 5 \mid (2 - a)$
  - $[2]_R = \{\ldots, -3, 2, 7, 12, \ldots\}$
  - $[1]_R = \{\ldots, -4, 1, 6, 11, \ldots\}$

# Equivalence Relations

- Let $R_n$ be a relation on the set of strings built with $\{0,1\}$.

  For any two strings s and t,

  $(s, t) \in R_n$ if s = t,

  or $l(s), l(t) \geq n$ and $s[1..n] = t[1..n]$

  ($l(s)$: length of s; $s[1..n]$: first n bits of s)

- $(01,01) \in R_3$, $(11,10) \notin R_3$

  $(101,101) \in R_3$, $(101,110) \notin R_3$

  $(0111,0110) \in R_3$, $(1101,1011) \notin R_3$

  $(01001,010111000) \in R_3$, $(1100,10011111) \notin R_3$

# Equivalence Relations

- Let $R_n$ be a relation on the set of strings built with $\{0,1\}$.

  For any two strings s and t,

  $(s,t) \in R_n$ if s = t,

  or $l(s), l(t) \geq n$ and $s[1..n] = t[1..n]$

- for all $a \in S$, since $a = a$, $(a,a) \in R_3$ (reflexive)
- if $(a,b) \in R_3$, either $a = b$ or $\mathrm{a}[1..3] = b[1..3]$
  thus $(b,a) \in R_3$ (symmetric)
- if $(a,b) \in R_3 \wedge (b,c) \in R_3$,
  either $a = b$ and $b = c$, then $a = c$
  or $a = b$ and $b[1..3] = c[1..3]$, then $a[1..3] = c[1..3]$
  or $a[1..3] = b[1..3]$ and $b = c$, then $a[1..3] = c[1..3]$
  or $a[1..3] = b[1..3]$ and $b[1..3] = c[1..3]$, then $a[1..3] = c[1..3]$
  (transitive)

# Equivalence Relations

- Let $R_n$ be a relation on the set of strings built with $\{0,1\}$.

  For any two strings s and t,

  $(s,t) \in R_n$ if s = t,

  or $l(s), l(t) \geq n$ and $s[1..n] = t[1..n]$

- $[0]_{R_3} = \{0\}, [1]_{R_3} = \{1\}, [00]_{R_3} = \{00\}, [01]_{R_3} = \{01\},$
  $[10]_{R_3} = \{10\}, [11]_{R_3} = \{11\}, [\varepsilon]_{R_3} = \{\varepsilon\}$

- $[000]_{R_3} = \{000,0000,0001,00000,00001,\ldots\}$
  $[001]_{R_3} = \{001,0010,0011,00100,00101,\ldots\}$
  $\vdots$
  $[111]_{R_3} = \{111,1110,1111,11100,11101,\ldots\}$

- $[\varepsilon]_{R_3} \cup [0]_{R_3} \cup \cdots \cup [111]_{R_3} = S$, the set of all strings

# Equivalence Relations

- A given set S can be decomposed into disjoint subsets $A_i$. For a family of sets $A = \{A_i | i \in I\}$ such that $A_i \cap A_j \neq \emptyset$, a given set S can be written as

$$S = A_1 \cup \ldots \cup A_n$$

S = {1, 2, 3, 4, 5, 6} can be written as $S = A_1 \cup A_2 \cup A_3$ where

$$A_1 = \{1, 2, 3\}, A_1 = \{4, 5\}, A_1 = \{6\}$$

- Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partititon of S.

If there is a partition of S, then there is an equivalence relation that has $A_i$ as its equivalence classes.

R = {(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1),
(4, 4), (4, 5), (5, 4), (5, 5),
(6, 6)}

# Partial Order

Definition : A relation R on a set A is called a partial order if it's reflexive, antisymmetric, and transitive. A set S together with a partial order R is called partially ordered set or poset, (S, R)

- Consider 'greater than or equal' relation ($\geq$) defined on integers. ($\geq$) is a partial order ?
  - $\forall a \in \mathbb{Z}$, since $a \geq a, (a, a) \in$ ($\geq$) (reflexive)
  - $[(a, b) \in R] \rightarrow (a \geq b) \rightarrow (b \ngeq a$ if $a \neq b)$ (antisymmetric)
  - $[(a, b) \in R \wedge (b, c) \in R] \rightarrow (a \geq b \wedge b \geq c)$
    $\rightarrow (a \geq c) \rightarrow [(a, c) \in R]$ (transitive)

# Partial Order

Definition : A relation R on a set A is called a partial order if it's reflexive, antisymmetric, and transitive. A set S together with a partial order R is called partially ordered set or poset, (S, R)

- Consider a relation R on integers such that $(a, b) \in R$ if a - b is a non-negative integer. R is a partial order ?
  - $\forall a \in \mathbb{Z}$, since $a - a = 0, (a, a) \in R$ (reflexive)
  - $[(a, b) \in R] \rightarrow$ ($a - b$ is a non-negative integer)
    $\rightarrow$ ($\mathrm{b} - a$ is a negative integer if $a \neq b$) (antisymmetric)
  - $[(a, b) \in R \wedge (b, c) \in R] \rightarrow$ ($a - b$ and $b - c$ are non-negative integer)
    $\rightarrow$ ($a - c$ is a non-negative integer)
    $\rightarrow [(a, c) \in R]$ (transitive)

# Partial Order

Definition : The elements a and b of a poset (S, R) are called comparable if either aRb or bRa.

- Consider the poset $(\mathbb{Z}^+, '|')$.
  - Since $3|9$, 3 and 9 are comparable.
  - Since $7\nmid 5$ or $5\nmid 7$, 5 and 7 are not comparable

Definition : If every pair of elements in S are comparable, then R is called total order. (S, R) is called totally ordered set.

- the poset $(\mathbb{Z}^+, '|')$ is not totally ordered set.
- the poset $(\mathbb{Z}^+, \leq)$ is a totally ordered set.

# Partial Order

Definition : Consider a poset (S, R). An element a is called maximal if there is no $b \in S$ such that aRb. An element a is called minimal if there is no $b \in S$ such that bRa.

- Consider the poset (S, '|') where S = {2, 4, 5, 10, 12, 15, 20, 30}
  - maximal elements of (S, '|') {12, 20, 30}
  - minimal elements of (S, '|') {2, 5}

Definition : An element a is called the greatest element if bRa for all $b \in S$. An element a is called the least element if aRb for all $b \in S$.

- Consider the power set of a given set S.
  - ∅ is the least element of (P(S),⊆) since $\emptyset \subseteq T$ for any $T \in P(S)$
  - S is the greatest element of (P(S),⊆) since $T \subseteq S$ for any $T \in P(S)$

# Partial Order

- Let A = {0, 1, 2}, B = AxA, R be a relation defined on B such that

  $((a,b),(c,d)) \in R$ if $a < c$ or
  $a = c$ and $b \leq d$

  - $((0,1),(1,0)) \in R$ since $a < c$
  - $((0,1),(0,2)) \in R$ since $a = c$ and $b \leq d$

- R is partial order relation ?
  - for all $(a,b) \in B$, since $a = a$ and $b \leq b$, $((a,b),(a,b)) \in R$
  - for all $((a,b),(c,d)) \in R$ such that $(a,b) \neq (c,d)$
    either $a < c$, then $((c,d),(a,b)) \notin R$
    or $a = c$ and $b < d$, then $((c,d),(a,b)) \notin R$
  - for all $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$,
    either $a < c$ and $c < e$, then $a < e$, $((a,b),(e,f)) \in R$
    or $a < c$, and $c = e$ and $d \leq f$, then $a < e$, $((a,b),(e,f)) \in R$
    or $a = c$, and $c < e$, then $a < e$, $((a,b),(e,f)) \in R$
    or $a = c$ and $b \leq d$, and $c = e$ and $d \leq f$, then $a = e$ and $b \leq f$,
    $((a,b),(e,f)) \in R$

# Partial Order

- Let A = {0, 1, 2}, B = AxA, R be a relation defined on B such that

  $\big((a,b),(c,d)\big) \in R$ if $a < c$ or
  $a = c$ and $b \leq d$

  - $\big((0,1),(1,0)\big) \in R$ since $a < c$
  - $\big((0,1),(0,2)\big) \in R$ since $a = c$ and $b \leq d$

- R is partial order relation ?
  - Is there a least element ?

    (0, 0)
  - Is there a greatest element ?

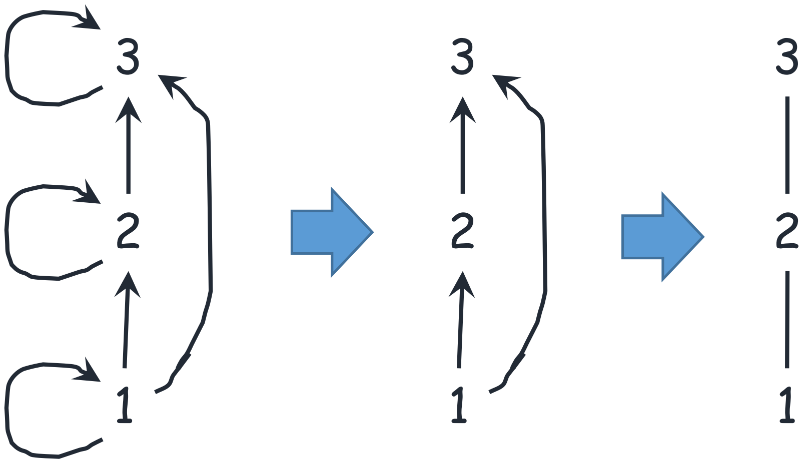
    (2, 2)
  - Is it total order ?

    for all $a, b \in B$, $(a,b) \in R$ or $(b,a) \in R$
  - How many elements are in R ?

    $(0,0)R(0,1)R(0,2)R(1,0)R(1,1)R(1,2)R(2,0)R(2,1)R(2,2)$

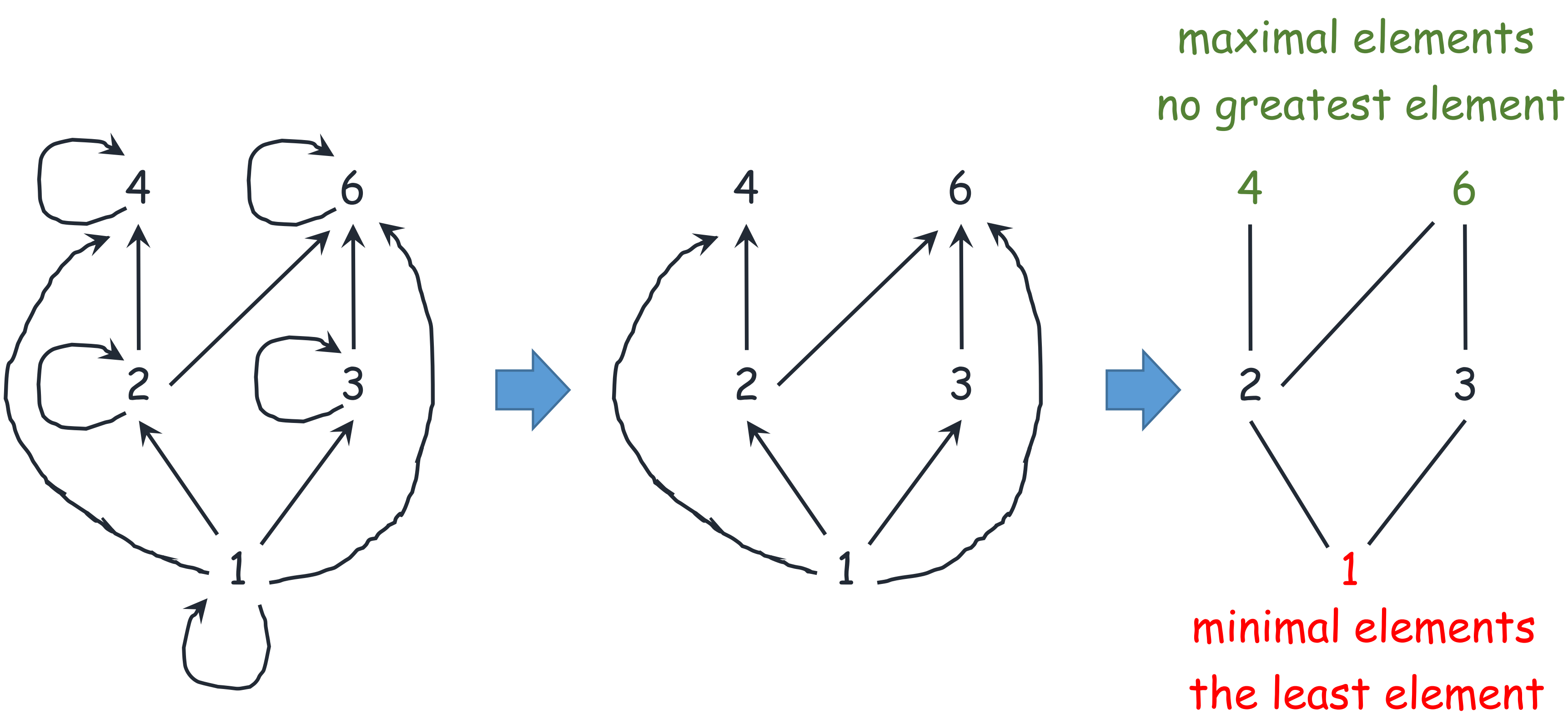
# Hasse Diagram

- a type of directed graph used to represent finite posets.
- consider the poset $(\{1, 2, 3\}, \leq)$: $(x, y) \in (\leq)$ if $x \leq y$
- consider the elements of the set as vertices
- if $(x, y) \in (\leq)$, draw a line from x to y



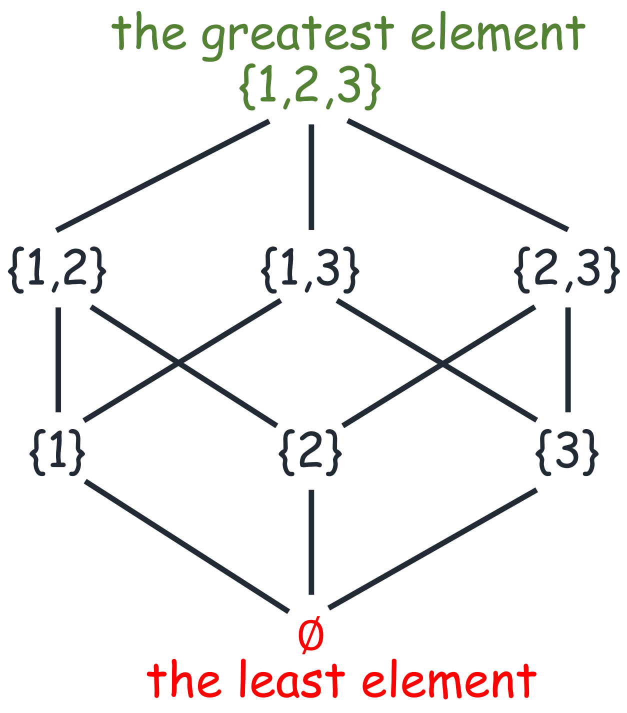
# Hasse Diagram

- a type of directed graph used to represent finite posets.
- consider the poset $(\{1, 2, 3, 4, 6\}, R)$: $(x, y) \in R$ if $x$ divides $y$



# Hasse Diagram

- a type of directed graph used to represent finite posets.
- consider the poset $(\{1,2,3\}, R)$: $(X,Y) \in R$ if $X \subseteq Y$



# Partial Order

Definition : Consider a poset (S, R). If there is an element $u \in S$ such that $aRu$ for all $a \in A$ , then u is called an upper bound of A. If there is an element $v \in S$ such that $vRa$ for all $a \in A$ , then v is called an lower bound of A.

- Consider the poset $(\mathbb{Z}^+, '|')$
  - for the set A = {3, 9, 12};

    if $u|3$, $u|9$, $u|12$, then u is a lower bound : 1 and 3

    if $3|v$, $9|v$, $12|v$, then v is an upper bound : 36, 72, …
  - for the set B = {1, 2, 4, 5, 10};

    if $u|1$, $u|2$, $u|4$, $u|5$, $u|10$, then u is a lower bound : 1

    if $1|v$, $2|v$, $4|v$, $5|v$, $10|v$, then v is an upper bound : 20, 40, …

# Partial Order

Definition : Consider a poset (S, R). If there is an element $u \in S$ such that $aRu$ for all $a \in A$ , then u is called an upper bound of A. If there is an element $v \in S$ such that $vRa$ for all $a \in A$ , then v is called an lower bound of A.

- Consider the poset $(P(S), \subseteq)$ where S = {1, 2, 3, 4}
  - for the set A = {{1}, {2}, {1,2}};

    if U⊆{1}, U⊆{2},U⊆{1,2}, then U is a lower bound : ∅

    if {1}⊆V, {2}⊆V, {1,2}⊆V, then V is an upper bound :

    {1,2}, {1,2,3}, {1,2,4}, {1,2,3,4}

# Topological Sorting

Definition : Topological sorting of n elements from a poset (S, R) is $s_1 s_2 \dots s_n$ such that there is no $(s_i, s_j) \in R$ where $j < i$

- Consider the poset (S, '|') where S = {2, 15, 8, 3, 6, 20}
  - 2, 3, 6, 8, 15, 20
  - 3, 2, 8, 6, 15, 20
  - 3, 2, 6, 8, 20, 15
  - 3, 6, 2, 8, 20, 15 is not, i.e. (2, 6) ∈ '|' since $2|6$, but 6 comes before 2 in the sorting.

# Topological Sorting

Definition : Topological sorting of n elements from a poset (S, R) is $s_1 s_2 \ldots s_n$ such that there is no $(s_i, s_j) \in R$ where $j < i$

- Every finite nonempty poset (S, R) has at least one minimal element.
  - Pick an element $a_0 \in S$. If $a_0$ is not minimal, then there should be an element $a_1 \in S$ such that $a_1 R a_0$.
  - If $a_1$ is not minimal, then there should be an element $a_2 \in S$ such that $a_2 R a_1$.

$$\vdots$$

  - Since there are only finite number of elements, there should be an element $a_n$ that is minimal.

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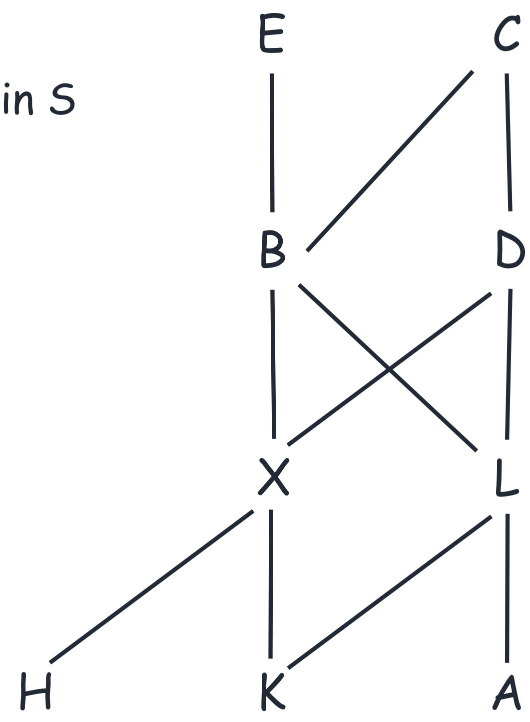
input : a finite poset (S, R)
output : topological sorting of elements in S

```
initialize an empty queue Q
while S ≠ ∅
        a = a minimal element of S
        S = S – {a}
        add a to Q
return Q
```



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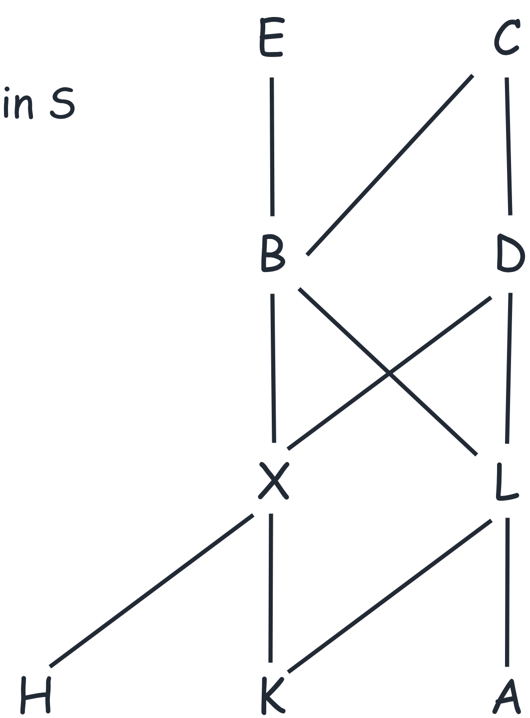


Q :

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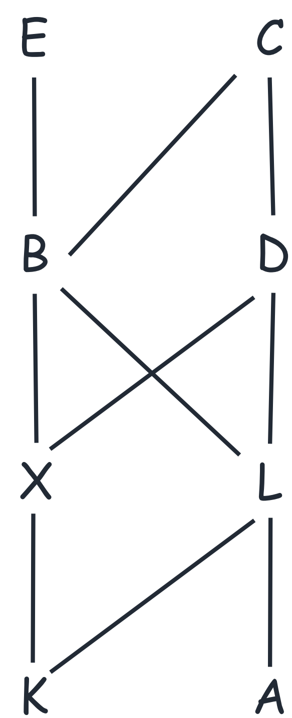


Q : H

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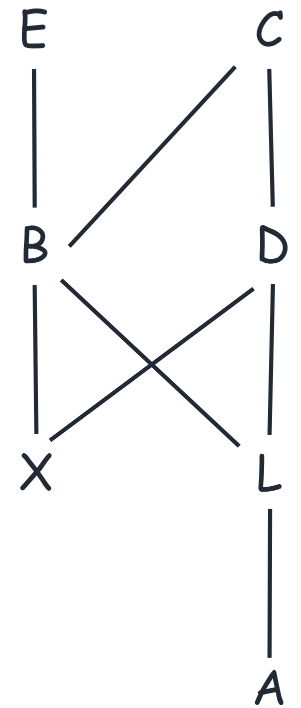


Q : H K

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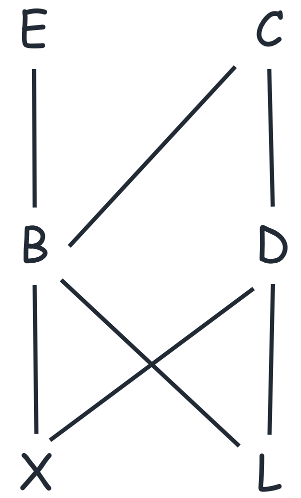


Q : H K A

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    add a to Q
return Q

Q : H K A L X B E D C