# Mathematical Induction

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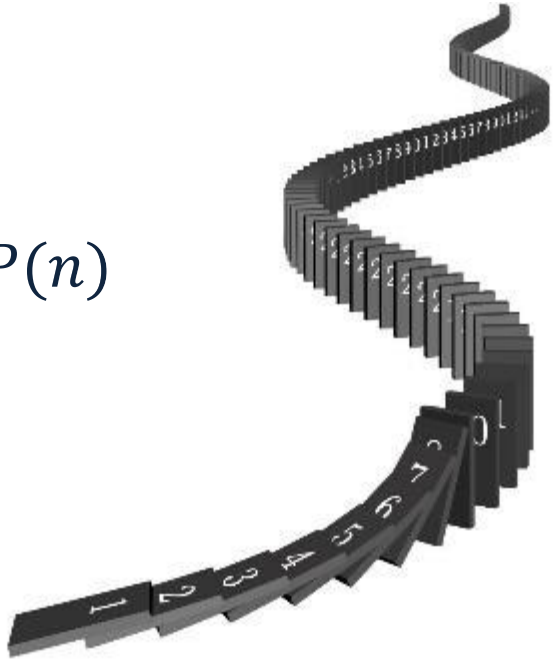
# Definition

- To prove P(n) is true for all positive integers n,
  - verify that P(1) is true (Basic Step)
  - prove that the implication $P(k) \rightarrow P(k+1)$ for all $k \in Z^{+}$ (Inductive Step)

$$[P(1) \wedge \ \forall k P(k) \rightarrow P(k+1)] \rightarrow \forall n P(n)$$



# Proofs

- Prove that $\forall x \in Z^+, x^3 - x$ is divisible by 3

Basic Step P(1) : $1^3 - 1 = 0$ is divisible by 3

Inductive Step $P(k) \rightarrow P(k+1)$

assume that P(k) is true, i.e $k^3 - k$ is divisible by 3

$$
\begin{aligned}
[k^3 - k = 3a, \exists a \in \mathbb{Z}] &\rightarrow (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 \\
&\rightarrow (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k - k \\
&\rightarrow (k+1)^3 - (k+1) = k^3 - k + 3k^2 + 3k \\
&\rightarrow (k+1)^3 - (k+1) = k^3 - k + 3(k^2 + k) \\
&\rightarrow (k+1)^3 - (k+1) = 3a + 3b, \exists a, b \in \mathbb{Z} \\
&\rightarrow (k+1)^3 - (k+1) \text{ is divisible by 3}
\end{aligned}
$$

# Proofs

- Prove that $\forall n \in N$, $7^{n+2} + 8^{2n+1}$ is divisible by 57

Basic Step P(0) : $7^2 + 8 = 57$ is divisible by 57

Inductive Step $P(k) \to P(k+1)$

assume that P(k) is true, i.e $7^{k+2} + 8^{2k+1}$ is divisible by 57

$$[7^{k+2} + 8^{2k+1} = 57a, \exists a \in \mathbb{Z}] \to 7^{k+3} + 8^{2k+3} = 7.7^{k+2} + 64.8^{2k+1}$$

$$\to 7^{k+3} + 8^{2k+3} = 7.7^{k+2} + 7.8^{2k+1} + 57.8^{2k+1}$$

$$\to 7^{k+3} + 8^{2k+3} = 7(7^{k+2} + 8^{2k+1}) + 57.8^{2k+1}$$

$$\to 7^{k+3} + 8^{2k+3} = 57a + 57b, \exists a, b \in \mathbb{Z}$$

$\to 7^{k+3} + 8^{2k+3}$ is divisible by 57

# Proofs

- Prove that if $\forall n \in Z^+$, then $1 + 2 + \ldots + n = n.(n+1)/2$

Basic Step P(1) : $1 = 1.2/2$

Inductive Step $P(k) \rightarrow P(k+1)$

assume that P(k) is true, i.e $1 + 2 + \ldots + k = k.(k+1)/2$

$$[1 + 2 + \ldots + k = k.(k+1)/2\ ] \rightarrow \left[1 + 2 + \ldots + (k+1) = k.\frac{k+1}{2} + k + 1\right]$$

$$\rightarrow \left[1 + 2 + \ldots + (k+1) = \frac{k(k+1) + 2(k+1)}{2}\right]$$

$$\rightarrow \left[1 + 2 + \ldots + (k+1) = \frac{(k+1)(k+2)}{2}\right]$$

# Proofs

Conjecture a formula for the sum of the first n positive odd integers, then prove your conjecture using mathematical induction

- 1 = 1 $1^2$ | 1 + 3 = 4 $2^2$ | 1 + 3 + 5 = 9 $3^2$ | 1 + 3 + 5 + 9 = 16 $4^2$

  1 + 3 + . . . + (2n – 1) = $n^2$

Basic Step P(1) : $1 = 1^2$

Inductive Step $P(k) \to P(k+1)$

assume that P(k) is true, i.e $1 + 2 + \ldots + (2k-1) = k^2$

$$[1 + 2 + \ldots + (2k-1) = k^2] \to [1 + 2 + \ldots + (2k-1) + (2k+1) = k^2 + 2k + 1]$$

$$\to [1 + 2 + \ldots + (2k-1) + (2k+1) = (k+1)^2]$$

# Proofs

- Prove that if $\forall n \in \mathbb{N}$, then $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

Basic Step P(1) : $1 = 2^{0+1} - 1$

Inductive Step $P(k) \to P(k+1)$

assume that P(k) is true, i.e $1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$

$[1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1\ ] \to [1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} -$

# Proofs

- Prove that for every integer $n \geq 4,\ 2^n < n!$

Basic Step P(4) : $2^4 = 16 < 4! = 24$

Inductive Step $P(k) \rightarrow P(k+1)$

assume that P(k) is true, i.e $2^k < k!$

$$[2^k < k!\ ] \rightarrow [2^{k+1} = 2.2^k < 2.k!\ ] \rightarrow \left[2^{k+1} < 2.k! < (k+1).k!\right]$$

$$\rightarrow [2^{k+1} < (k+1)!\ ]$$

# Proofs

$$H_j = 1 + \frac{1}{2} + \ldots + \frac{1}{j}$$

- Prove that $H_1 + H_2 + \ldots + H_n = (n+1)H_n - n$

Basic Step P(1) : $[H_1 \stackrel{?}{=} 2.H_1 - 1] \rightarrow [1 = 2 - 1]$

Inductive Step $P(k) \rightarrow P(k+1)$

assume that P(k) is true, i.e $H_1 + \ldots + H_k = (k+1)H_k - k$

$$[H_1 + \ldots + H_k = (k+1)H_k - k\,] \rightarrow [H_1 + \ldots + H_k + H_{k+1} = (k+1)H_k - k + H_{k+1}]$$

$$\rightarrow \left[H_1 + \ldots + H_{k+1} = (k+1)(H_k - \tfrac{1}{k+1} + \tfrac{1}{k+1}) - k + H_{k+1}\right]$$

$$\rightarrow \left[H_1 + \ldots + H_{k+1} = (k+1)(H_{k+1} - \tfrac{1}{k+1}) - k + H_{k+1}\right]$$

$$\rightarrow [H_1 + \ldots + H_{k+1} = (k+1)H_{k+1} - 1 - k + H_{k+1}]$$

$$\rightarrow [H_1 + \ldots + H_{k+1} = (k+2)H_{k+1} - (k+1)\,]$$

# Proofs

- For every integer $n \geq 14$, $n$ can be written as a sum of 3's and 8's

19 = 3 + 8 + 8 = 1.3 + 2.8
20 = 3 + 3 + 3 + 3 + 8 = 4.3 + 1.8

Basic Step P(4) : $14 = 2.3 + 1.8$

Inductive Step $P(k) \rightarrow P(k+1)$

assume that P(k) is true, i.e $k = a.3 + b.8, \exists a, b \in N$

if $b > 0$, $k+1 = a.3 + b.8 + 1$
$k+1 = a.3 + (b-1).8 + 8 + 1$
$k+1 = (a+3).3 + (b-1).8$

$P(k-8)$

if $b = 0$, $k+1 = a.3 + 1$
$k+1 = (a-5).3 + 15 + 1$
$k+1 = (a-5).3 + 2.8$

$P(k-15)$

$[P(k-8) \wedge P(k-15)] \rightarrow P(k+1)$

# Strong Induction

- To prove P(n) is true for all positive integers n,
  - verify that P(1) is true (Basic Step)
  - prove that the implication

    $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$

    for all $k \in Z^{+}$ (Inductive Step)

# Strong Induction

- Prove that for every integer $n \geq 2$, $n$ can be written as the product of primes

Basic Step P(2) is true, i.e. 2 can be written as the product of primes

Inductive Step $[P(1) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$

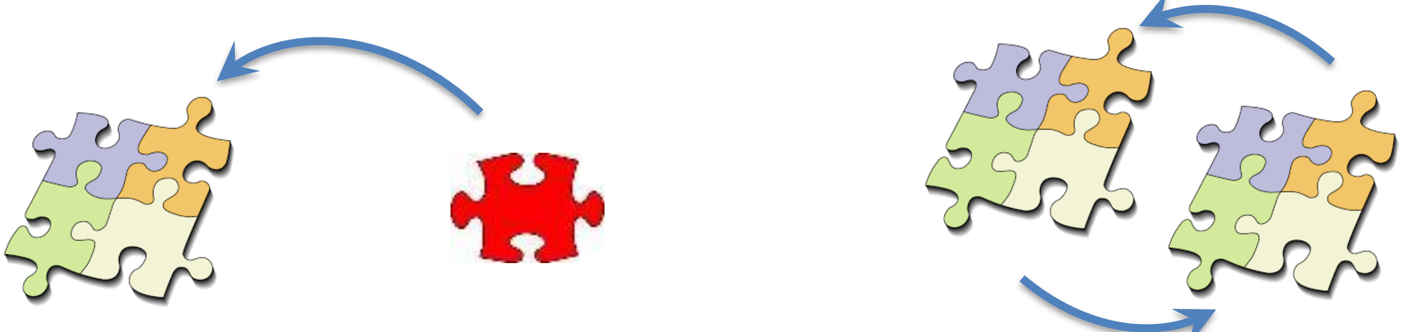
assume that P(i) is true for all i such that $2 \leq i \leq k$, i.e i can be written as the product of primes, then

if $(k+1)$ is prime, then $P(k+1)$ is true

if $(k+1)$ is composite, then $k+1 = a.b$, where $2 \leq a \leq b < k+1$. Since $a, b < k+1$, $P(a)$ and $P(b)$ are true from the assumption, i.e. a and b can be written as the product of primes. Thus, $k+1 = a.b$ can also be written as the product of primes.

# Strong Induction

- Consider a puzzle. How do we assemble a puzzle ?

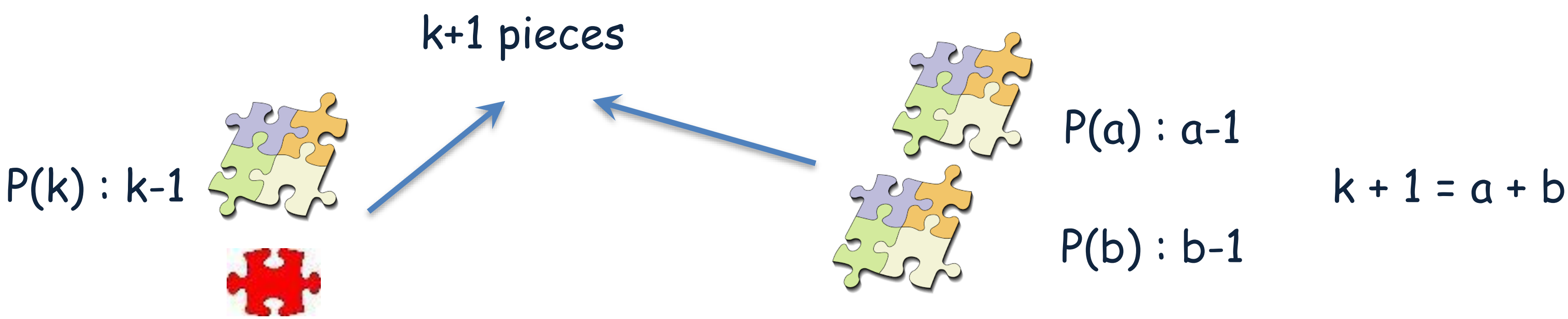


- Show that no matter which move we make, n-1 noves required to assemble a puzzle with n pieces.

Basic Step  P(1) is true, i.e. no move required for just 1 piece

Inductive Step $[P(1) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$

assume that P(i) is true for all i such that $2 \leq i \leq k$, i.e a puzzle with i pieces can be assembled with i-1 moves



# Strong Induction

- Prove that for every integer $n \geq 3$, $\mathrm{F}(n) > \alpha^{n-2}$ where $\alpha = (1 + \sqrt{5})/2$

Fibonacci sequence : $\mathrm{F(1)} = 1, \mathrm{F(2)} = 1,$ and $\mathrm{F(n)} = \mathrm{F(n-1)} + \mathrm{F(n-2)}$

Basic Step P(3) : $\mathrm{F}(3) = 2 > \alpha^{3-2} = (1 + \sqrt{5})/2$

Inductive Step $[P(1) \wedge \ \ldots \wedge P(k)] \rightarrow P(k+1)$

assume that P(i) is true for all i such that $2 \leq i \leq k$, i.e $\mathrm{F}(i) > \alpha^{i-2}$

for $P(k+1)$ : $\mathrm{F}(k+1) = \mathrm{F}(k) + \mathrm{F}(k-1) > \alpha^{i-2} + \alpha^{i-3}$

$$= \alpha . \alpha^{i-3} + \alpha^{i-3}$$

$$= (\alpha + 1) . \alpha^{i-3} = \alpha^2 . \alpha^{i-3}$$

$$\mathrm{F}(k+1) > \alpha^{i-1}$$

$\alpha = \frac{1+\sqrt{5}}{2}$ is a solution of the equation $\alpha^2 - \alpha - 1 = 0$. Thus, $\alpha^2 = \alpha + 1$

# Proofs

Conjecture a formula for the sum of the squares of the first n terms in Fibonacci sequence, then prove your conjecture using mathematical induction

- $F(1)^2 = 1, F(1)^2 + F(2)^2 = 2, F(1)^2 + F(2)^2 + F(3)^2 = 6,$
  $F(1)^2 + F(2)^2 + F(3)^2 + F(4)^2 = 15,$
  $F(1)^2 + F(2)^2 + F(3)^2 + F(4)^2 + F(5)^2 = 40,$

  1.1 1.2 2.3 3.5 5.8

- $\sum_{j=1}^{n} F(j)^2 = F(n).F(n+1)$, where $n \geq 2$

  Basic Step P(2) :$F(1)^2 + F(2)^2 = 2 = \mathrm{F}(2).\mathrm{F}(3)$

  Inductive Step $P(k) \rightarrow P(k+1)$

assume that P(i) is true for all i such that $2 \leq i \leq k$, i.e. $\sum_{j=1}^{i} F(j)^2 = F(i).F(i+1)$

for $P(k+1)$ : $F(1)^2 + \cdots + F(k)^2 + F(k+1)^2 = F(k).F(k+1) + F(k+1)^2$

$$= F(k+1)(F(k) + F(k+1))$$

$$= F(k+1)F(k+2)$$

# Proofs

Conjecture a formula for the sum of the squares of the first n terms in Fibonacci sequence, then prove your conjecture using mathematical induction

- $F(1)^2 = 1$, $F(1)^2 + F(2)^2 = 2$, $F(1)^2 + F(2)^2 + F(3)^2 = 6$,
  $F(1)^2 + F(2)^2 + F(3)^2 + F(4)^2 = 15$,
  $F(1)^2 + F(2)^2 + F(3)^2 + F(4)^2 + F(5)^2 = 40$,

  1.1 1.2 2.3 3.5 5.8

- $\sum_{j=1}^{n} F(j)^2 = F(n).F(n+1)$, where $n \geq 2$

  Basic Step P(2) :$F(1)^2 + F(2)^2 = 2 = \mathrm{F}(2).\mathrm{F}(3)$

  Inductive Step $P(k) \rightarrow P(k+1)$

assume that P(i) is true for all i such that $2 \leq i \leq k$, i.e. $\sum_{j=1}^{i} F(j)^2 = F(i).F(i+1)$

for $P(k+1)$ : $F(1)^2 + \cdots + F(k)^2 + F(k+1)^2 = F(k).F(k+1) + F(k+1)^2$

$$= F(k+1)(F(k) + F(k+1))$$

$$= F(k+1)F(k+2)$$

# Proofs

- Let's recursively define a set :

Basic Step $3 \in S$

Inductive Step if $x \in S$ and $y \in S$, then $x + y \in S$

$3 + 3 = 6 \in S, 3 + 6 = 9 \in S, \ldots$

- If A is the set of all positive integers that are divisible by 3, $A \stackrel{?}{=} S$

(A⊆S : every positive integer that is divisible by 3 is in S)

P(n) : '3n belongs to S'
Basic Step $3.1 = 3 \in S$
Inductive Step $P(k) \rightarrow P(k+1)$
assume that $P(k)$ is true, i.e. 3k belongs to S → 3k+3 also belongs to S

(S⊆A : every element of S is divisible by 3)

P(n) : '$n \in S$ is divisible by 3'
Basic Step 3 | 3
Inductive Step $[P(1) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$
assume that $P(i)$ is true for all $1 \leq i \leq k$, then $k + 1 = x + y$ where $x, y \leq k$
Since P(x) and P(y) assumed to be true, P(k+1) is also true