Analysis of Algorithm Efficiency

Murat Osmanoglu

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  - the RAM model of computation
  - the asymptotic analysis

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  - each such instruction (and each memory access) takes a constant amount of time

Analysis Framework

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  - for a spell-checking algorithm, if the algorithm works the individual characters, it will be the number of characters, if it works on words, it will be the number of words
- for some algorithms it might be the magnitute of a single input
  - for an algorithm checking primality of a given positive integer n, it will be $\lfloor \log n \rfloor + 1$

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    (we can answer such question without knowing them)
  - focus on rate of growth of the function $T(n)$

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Order of Growth (Rate of Growth)

- For small inputs, a difference in running times can be ignored
  (it does not actually distinguish efficient algorithms from inefficient ones)
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- Growth rates of common functions measured in nanoseconds (assume each operation takes one nanosecond)

n $f(n)$	$\lg n$	n	$n \lg n$	n^2	2^n	$n!$
10	0.003 μs	0.01 μs	0.033 μs	0.1 μs	1 μs	3.63 ms
20	0.004 μs	0.02 μs	0.086 μs	0.4 μs	1 ms	77.1 years
30	0.005 μs	0.03 μs	0.147 μs	0.9 μs	1 sec	8.4×10^{15} yrs
40	0.005 μs	0.04 μs	0.213 μs	1.6 μs	18.3 min	
50	0.006 μs	0.05 μs	0.282 μs	2.5 μs	13 days	
100	0.007 μs	0.1 μs	0.644 μs	10 μs	4×10^{13} yrs	
1,000	0.010 μs	1.00 μs	9.966 μs	1 ms		
10,000	0.013 μs	10 μs	130 μs	100 ms		
100,000	0.017 μs	0.10 ms	1.67 ms	10 sec		
1,000,000	0.020 μs	1 ms	19.93 ms	16.7 min		
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- by analyzing the order of growth of the function $T(n)$ that counts the algorithm's basic operation (simply considering the leading term of the function), we can evaluate whether a given algorithm is practical for a problem of a given size

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Worst-Case, Best-Case, Average-Case Analysis

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Linear-Search(list,x)

input : $\{a_1, a_2, \ldots, a_n; x\}$
output: location

for i = 1 to n
    if $x = a_i$
    return i
return 0

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Worst Case :

- consider the worst-case input of size n for which the algorithm runs the longest among all possible inputs of same size

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- consider the worst-case input of size n for which the algorithm runs the longest among all possible inputs of same size

  (the element x matches the last one in the list, or the list does not contain the element x)

- $T(n) = n + 3$

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(if, return i, return loc: 1 op)

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- consider the best-case input of size n for which the algorithm runs the fastest among all possible inputs of same size

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- $T(n) = 2$

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Average Case :

- neither the worst-case nor the best-case analysis gives us the necessary information about how the algorithm behaves on a random input
- it's the expected value for the number of operations

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- if $x = a_1$, then the algorithm terminates after 2 operations

  if $x = a_2$, then the algorithm terminates after 3 operations

  ⋮

  if $x = a_i$, then the algorithm terminates after $i + 1$ operation

  ⋮

  if $x = a_n$, then the algorithm terminates after $n + 1$ operations

  if $x \notin L$, then the algorithm terminates after $n + 1$ operations

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  ⋮

  if $x = a_i$, then the algorithm terminates after $i + 1$ operation

  ⋮

  if $x = a_n$, then the algorithm terminates after $n + 1$ operations

  if$x \notin L$, then the algorithm terminates after $n + 1$ operations

Average Case :

- let p be the probability that $x \in L$, and $q = 1 - p$ be the probability that $x \notin L$

Analysis Framework

Worst-Case, Best-Case, Average-Case Analysis

Linear-Search(list,x)

input : $\{a_1, a_2, \ldots, a_n; x\}$
output: location

for i = 1 to n —— n steps
  if $x = a_i$
  return i
return loc

1 op

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Average Case :

- let p be the probability that $x \in L$, and $q = 1 - p$ be the probability that $x \notin L$
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$$E(X) = \sum p(s).X(s)$$

Analysis Framework

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$$E(X) = \sum p(s).X(s)$$

$$= 2.\frac{p}{n} + 3.\frac{p}{n} + \ldots + (n+1).\frac{p}{n} + (n+1).q = p\frac{(n+3)}{2} + q.(n+1)$$

Analysis Framework

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- for $p = 1$ and $q = 0$
  $E(X) = (n+3)/2$
- for $p = 0$ and $q = 1$
  $E(X) = n + 1$
- for $p = q = 1/2$
  $E(X) = (3n+5)/4$

Analysis Framework

Worst-Case, Best-Case, Average-Case Analysis

Linear-Search(list,x)

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  …minates

  …nates

- average-case analysis is more difficult than worst-case and best-case analysis
- applying the corresponding values to formula is easy, but probabilistic assumption for each particular case is hard to verify
- mostly deal with worst-case analysis

Average Case :

- let p be the pr… $q = 1 - p$ be th…
- for each element a_i, the probability that $x = a_i$ is p/n
- the expected value for the number of operations

$$E(X) = \sum p(s).X(s)$$
$$= 2.\frac{p}{n} + 3.\frac{p}{n} + \ldots + (n+1).\frac{p}{n} + (n+1).q = p\frac{(n+3)}{2} + q.(n+1)$$

- …$= 1$ and $q = 0$
  $= (n+3)/2$
- for $p = 0$ and $q = 1$
  $E(X) = n + 1$
- for $p = q = 1/2$
  $E(X) = (3n+5)/4$

Asymptotic Notations

- for the algorithm’s efficiency, we focus on the order of growth of the function that counts the algorithm’s basic operations

Asymptotic Notations

- for the algorithm's efficiency, we focus on the order of growth of the function that counts the algorithm's basic operations

- to compare and rank such orders of growth, three common tools will be employed:

  - O (big-oh), asymptotic upper bound

  - Ω (big-omega), asymptotic lower bound

  - Θ (big-theta), asymptotic tight bound

Asymptotic Notations

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$$24n^3 \in \Omega(n^2), 3n(n-1) \in \Omega(n^2), 27n + 100 \notin \Omega(n^2)$$

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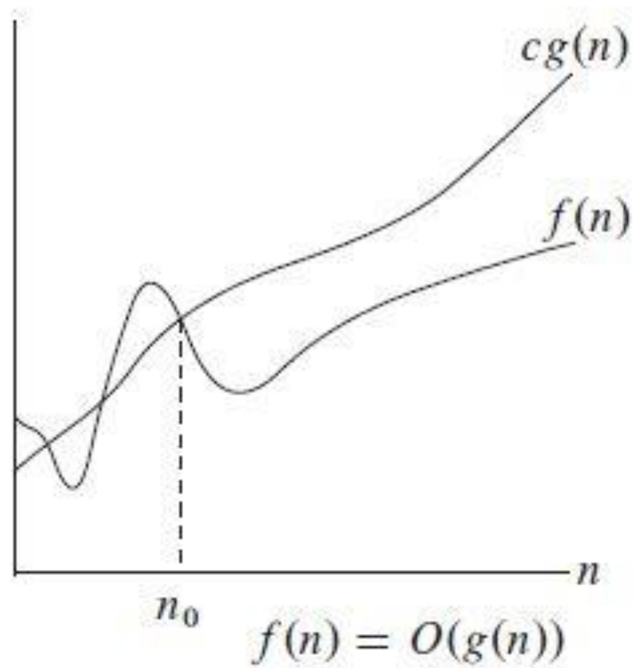
- Θ(g(n)) is the class of all functions with the same order of growth as g(n)

$$24n^2 + 17n \in \Theta(n^2), n^2 + 17\log n \in \Theta(n^2), 27n + 100 \notin \Theta(n^2), n^3 \notin \Theta(n^2)$$

Big-Oh Notation

Definition : Let $f, g : \mathbb{Z}^+ \to \mathbb{R}$ be two functions. If there are constants C and n_0 such that $|f(n)| \leq C.|g(n)|$ for all $n \in \mathbb{Z}$ where $n \geq n_0$, we say that f is big-oh of g,

$$f(n) = O(g(n))$$



Big-Omega Notation

Definition : Let $f, g : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be two functions. If there are constants C and n_0 such that $|f(n)| \geq C.|g(n)|$ for all $n \in \mathbb{Z}$ where $n \geq n_0$, we say that f is big-omega of g,

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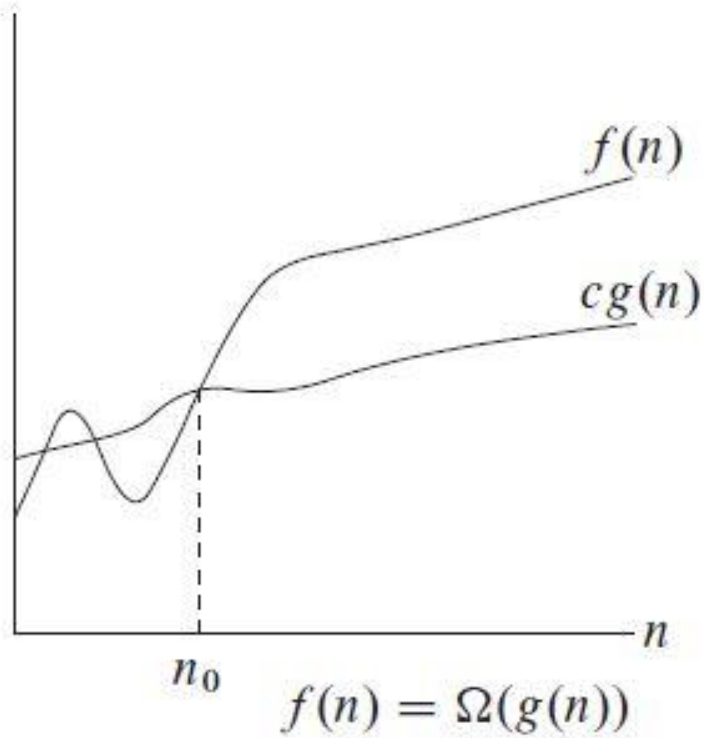
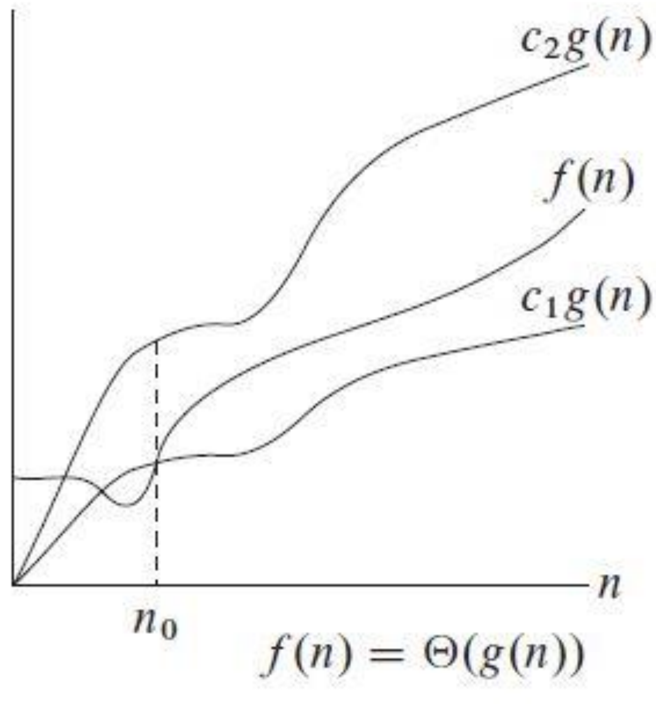


Big-Theta Notation

Definition : Let $f, g : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be two functions. If there are constants C_1, C_2, and n_0 such that $C_1.|g(n)| \leq |f(n)| \leq C_2.|g(n)|$ for all $n \in \mathbb{Z}$ where $n \geq n_0$, we say that f is big-theta of g,

$$f(n) = \Theta(g(n))$$



Big-Oh Notation

- $f, g : \mathbb{Z}^+ \to \mathbb{R}$, $f(n) = 5n$ and $g(n) = n^2$.

Big-Oh Notation

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  - $f(1) = 5, f(2) = 10, f(3) = 15, f(4) = 20, f(5) = 25, \ldots$
    $g(1) = 1, g(2) = 4, \; g(3) = 9, \;\; g(4) = 16, g(5) = 25, \ldots$

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  - for $n \geq 5, n^2 \geq 5n \rightarrow |f(n)| \leq |g(n)|$
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    $|f(n)| \leq C.|g(n)|$ for all $n \geq n_0$. Thus, $f(n) = O(g(n))$.

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  - for $n \geq 5$, $n^2 \geq 5n \to |f(n)| \leq |g(n)|$
  - for $C = 1$ and $n_0 = 5$,

    $|f(n)| \leq C.|g(n)|$ for all $n \geq n_0$. Thus, $f(n) = O(g(n))$.
  - C and n_0 don't have to be unique

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$$\leq 5n^2 + 3n^2 + n^2$$

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$$\leq 5n^2 + 3n^2 + n^2 = 9n^2$$

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  for $C = 9$ and $n_0 = 1$,

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for $C = 1$ and $n_0 = 1$,

$|g(n)| \leq C.|f(n)|$ for all $n \geq n_0$. Thus, $g(n) = O(f(n))$.

Big-Oh Notation

- $f, g : \mathbb{Z}^+ \to \mathbb{R}$, $f(n) = 7n^2$ and $g(n) = n^3$.

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  $|f(n)| = |7n^2| = 7n^2 \leq 7n^3 = 7|g(n)|$

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  for $C = 7$ and $n_0 = 1$,

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  for $C = 7$ and $n_0 = 1$,

  $|f(n)| \leq C.|g(n)|$ for all $n \geq n_0$. Thus, $f(n) = O(g(n))$.

  $|g(n)| = |n^3| = n^3 \leq C.7.n^2 = C.|f(n)|$

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  $|g(n)| = |n^3| = n^3 \leq C.7.n^2 = C.|f(n)| \to n \leq C.7$ for all $n \geq n_0$

  there cannot be any C and n_0 that satisfy this inequality.

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- $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$, $f(n) = a_t n^t + a_{t-1} n^{t-1} + \ldots + a_1 n + a_0$

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$$|f(n)| = |a_t n^t + a_{t-1} n^{t-1} + \ldots + a_1 n + a_0| \leq |a_t n^t| + \ldots + |a_1 n| + |a_0|$$
$$= |a_t| . n^t + \ldots + |a_1| . n + |a_0|$$

Big-Oh Notation

- $f : \mathbb{Z}^+ \to \mathbb{R}$, $f(n) = a_t n^t + a_{t-1} n^{t-1} + \ldots + a_1 n + a_0$

$$
\begin{aligned}
|f(n)| = |a_t n^t + a_{t-1} n^{t-1} + \ldots + a_1 n + a_0| &\leq |a_t n^t| + \ldots + |a_1 n| + |a_0| \\
&= |a_t|.\, n^t + \ldots + |a_1|.\, n + |a_0| \\
&\leq |a_t|.\, n^t + \ldots + |a_1|.\, n^t + |a_0|.\, n^t
\end{aligned}
$$

Big-Oh Notation

- $f:\mathbb{Z}^+ \to \mathbb{R}$, $f(n) = a_t n^t + a_{t-1}n^{t-1} + \ldots + a_1 n + a_0$

$$
\begin{aligned}
|f(n)| = |a_t n^t + a_{t-1}n^{t-1} + \ldots + a_1 n + a_0| &\leq |a_t n^t| + \ldots + |a_1 n| + |a_0| \\
&= |a_t|.n^t + \ldots + |a_1|.n + |a_0| \\
&\leq |a_t|.n^t + \ldots + |a_1|.n^t + |a_0|.n^t \\
&\leq (|a_t| + \ldots + |a_1| + |a_0|).n^t = C.|n^t|
\end{aligned}
$$

Big-Oh Notation

- $f : \mathbb{Z}^+ \to \mathbb{R},\ f(n) = a_t n^t + a_{t-1} n^{t-1} + \ldots + a_1 n + a_0$

$$\begin{aligned}|f(n)| = |a_t n^t + a_{t-1} n^{t-1} + \ldots + a_1 n + a_0| &\leq |a_t n^t| + \ldots + |a_1 n| + |a_0| \\ &= |a_t|.n^t + \ldots + |a_1|.n + |a_0| \\ &\leq |a_t|.n^t + \ldots + |a_1|.n^t + |a_0|.n^t \\ &\leq (|a_t| + \ldots + |a_1| + |a_0|).n^t = C.|n^t|\end{aligned}$$

for $C = |a_t| + \ldots + |a_1| + |a_0|$ and $n_0 = 1$,
$|f(n)| \leq C.|n^t|$ for all $n \geq n_0$. Thus, $f(n) = O(n^t)$

Big-Oh Notation

- $f : \mathbb{Z}^+ \to \mathbb{R}$, $f(n) = 1 + 2 + \ldots + n$

  $|f(n)| = |1 + 2 + \ldots + \mathrm{n}\ | = 1 + 2 + \ldots + n \leq n + n + \ldots + n = |n^2|$

  for $C = 1$ and $n_0 = 1$,

  $|f(n)| \leq C.|n^2|$ for all $n \geq n_0$. Thus, $f(n) = O(n^2)$

- $f : \mathbb{Z}^+ \to \mathbb{R}$, $f(n) = 1^2 + 2^2 + \ldots + n^2$

  $|f(n)| = |1^2 + 2^2 + \ldots + n^2| = 1^2 + 2^2 + \ldots + n^2 \leq n^2 + n^2 + \ldots + n^2 = |n^3|$

  for $C = 1$ and $n_0 = 1$,

  $|f(n)| \leq C.|n^3|$ for all $n \geq n_0$. Thus, $f(n) = O(n^3)$

- $f : \mathbb{Z}^+ \to \mathbb{R}$, $f(x) = 1^t + 2^t + \ldots + n^t$

  $|f(n)| = |1^t + 2^t + \ldots + n^t| = 1^t + 2^t + \ldots + n^t \leq n^t + n^t + \ldots + n^t = |n^{t+1}|$

  for $C = 1$ and $n_0 = 1$,

  $|f(n)| \leq C.|n^{t+1}|$ for all $n \geq n_0$. Thus, $f(n) = O(n^{t+1})$

Big-Oh Notation

- Basic Efficieny Classes

1	: constant
$\log n$	: logarithmic
n	: linear
$n \log n$	: linearithmic (loglinear)
n^2	: quadratic
n^t	: polynomial
2^n	: exponential
$n!$	: factorial

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O\big(g_2(n)\big)$

  $|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)|$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$
\begin{aligned}
|f_1(n) + f_2(n)| &\le |f_1(n)| + |f_2(n)| \\
&\le C_1|g_1(n)| + C_2|g_2(n)|
\end{aligned}
$$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O\left(g_2(n)\right)$

$$|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)|$$
$$\leq C_1|g_1(n)| + C_2|g_2(n)|$$
$$\leq C_1|g(n)| + C_2|g(n)| \quad \text{where} \quad g(n) = max\ \{g_1(n), g_2(n)\}$$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$|f_1(n) + f_2(n)| \le |f_1(n)| + |f_2(n)|$$
$$\le C_1|g_1(n)| + C_2|g_2(n)|$$
$$\le C_1|g(n)| + C_2|g(n)| \text{ where } g(n) = max\ \{g_1(n), g_2(n)\}$$
$$= (C_1+C_2)|g(n)|$$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)|$$
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$$\leq C_1|g(n)| + C_2|g(n)| \quad \text{where} \quad g(n) = max\ \{g_1(n), g_2(n)\}$$
$$= (C_1 + C_2)|g(n)|$$

$$f_1(n) + f_2(n) = O(max\ \{g_1(n), g_2(n)\})$$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O\left(g_2(n)\right)$

$$
\begin{aligned}
|f_1(n) + f_2(n)| &\le |f_1(n)| + |f_2(n)| \\
&\le C_1|g_1(n)| + C_2|g_2(n)| \\
&\le C_1|g(n)| + C_2|g(n)| \quad \text{where} \quad g(n) = max\ \{g_1(n), g_2(n)\} \\
&= (C_1+C_2)|g(n)|
\end{aligned}
$$

$$f_1(n) + f_2(n) = O(max\ \{g_1(n), g_2(n)\})$$

$$f_1(n). f_2(n) = O(g_1(n).g_2(n))$$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$
\begin{aligned}
|f_1(n) + f_2(n)| &\le |f_1(n)| + |f_2(n)| \\
&\le C_1|g_1(n)| + C_2|g_2(n)| \\
&\le C_1|g(n)| + C_2|g(n)| \quad \text{where} \quad g(n) = max\ \{g_1(n), g_2(n)\} \\
&= (C_1 + C_2)|g(n)|
\end{aligned}
$$

$$f_1(n) + f_2(n) = O(max\ \{g_1(n), g_2(n)\})$$

$$f_1(n). f_2(n) = O(g_1(n). g_2(n))$$

- $f(n) = (n+1)\log(n^2+1) + 3n^2$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$|f_1(n) + f_2(n)| \le |f_1(n)| + |f_2(n)|$$
$$\le C_1|g_1(n)| + C_2|g_2(n)|$$
$$\le C_1|g(n)| + C_2|g(n)| \quad \text{where} \quad g(n) = max\ \{g_1(n), g_2(n)\}$$
$$= (C_1 + C_2)|g(n)|$$

$$f_1(n) + f_2(n) = O(max\ \{g_1(n), g_2(n)\})$$

$$f_1(n).f_2(n) = O(g_1(n).g_2(n))$$

- $f(n) = (n+1)\log(n^2+1) + 3n^2$

$O(n)$ $\qquad\qquad$ $O(n^2)$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$
\begin{aligned}
|f_1(n) + f_2(n)| &\le |f_1(n)| + |f_2(n)| \\
&\le C_1|g_1(n)| + C_2|g_2(n)| \\
&\le C_1|g(n)| + C_2|g(n)| \quad \text{where} \quad g(n) = max\,\{g_1(n), g_2(n)\} \\
&= (C_1 + C_2)|g(n)|
\end{aligned}
$$

$$f_1(n) + f_2(n) = O(max\,\{g_1(n), g_2(n)\})$$

$$f_1(n).f_2(n) = O(g_1(n).g_2(n))$$

- $f(n) = (n+1)\log(n^2+1) + 3n^2$

$O(n)$ $\qquad$ $O(n^2)$

$$\log(n^2+1) \le \log(2n^2)$$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$|f_1(n) + f_2(n)| \le |f_1(n)| + |f_2(n)|$$
$$\le C_1|g_1(n)| + C_2|g_2(n)|$$
$$\le C_1|g(n)| + C_2|g(n)| \text{ where } g(n) = max\ \{g_1(n), g_2(n)\}$$
$$= (C_1 + C_2)|g(n)|$$

$$f_1(n) + f_2(n) = O(max\ \{g_1(n), g_2(n)\})$$

$$f_1(n).f_2(n) = O(g_1(n).g_2(n))$$

- $f(n) = (n+1)\log(n^2+1) + 3n^2$

$(n+1) \rightarrow O(n)$

$\log(n^2+1) \rightarrow$

$3n^2 \rightarrow O(n^2)$

$$\log(n^2+1) \le \log(2n^2)$$
$$= \log 2 + \log n^2$$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$|f_1(n) + f_2(n)| \le |f_1(n)| + |f_2(n)|$$
$$\le C_1|g_1(n)| + C_2|g_2(n)|$$
$$\le C_1|g(n)| + C_2|g(n)| \quad \text{where} \quad g(n) = max\,\{g_1(n), g_2(n)\}$$
$$= (C_1 + C_2)|g(n)|$$

$$f_1(n) + f_2(n) = O(max\,\{g_1(n), g_2(n)\})$$

$$f_1(n).f_2(n) = O(g_1(n).g_2(n))$$

- $f(n) = (n+1)\log(n^2+1) + 3n^2$

$O(n)$ $\qquad$ $O(n^2)$

$$\log(n^2+1) \le \log(2n^2)$$
$$= \log 2 + \log n^2$$
$$= \log 2 + 2\log n$$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)|$$
$$\leq C_1|g_1(n)| + C_2|g_2(n)|$$
$$\leq C_1|g(n)| + C_2|g(n)| \quad \text{where} \quad g(n) = max\ \{g_1(n), g_2(n)\}$$
$$= (C_1+C_2)|g(n)|$$

$$f_1(n) + f_2(n) = O(max\ \{g_1(n), g_2(n)\})$$

$$f_1(n).f_2(n) = O(g_1(n).g_2(n))$$

- $f(n) = (n+1)\log(n^2+1) + 3n^2$

$(n+1) \rightarrow O(n)$, $\log(n^2+1) \rightarrow O(\log n)$, $3n^2 \rightarrow O(n^2)$

$$\log(n^2+1) \leq \log(2n^2)$$
$$= \log 2 + \log n^2$$
$$= \log 2 + 2\log n$$
$$\leq 3\log n$$

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O\left(g_2(n)\right)$

$$
\begin{aligned}
|f_1(n) + f_2(n)| &\le |f_1(n)| + |f_2(n)| \\
&\le C_1|g_1(n)| + C_2|g_2(n)| \\
&\le C_1|g(n)| + C_2|g(n)| \quad \text{where} \quad g(n) = max\,\{g_1(n), g_2(n)\} \\
&= (C_1 + C_2)|g(n)|
\end{aligned}
$$

$$f_1(n) + f_2(n) = O(max\,\{g_1(n), g_2(n)\})$$

$$f_1(n).\,f_2(n) = O(g_1(n).g_2(n))$$

- $f(n) = (n+1)\log(n^2+1) + 3n^2$

$O(n)$ $\quad$ $O(\log n)$ $\quad$ $O(n^2)$

$$
\begin{aligned}
\log(n^2+1) &\le \log(2n^2) \\
&= \log 2 + \log n^2 \\
&= \log 2 + 2\log n \\
&\le 3\log n
\end{aligned}
$$

$$f(n) = O(n^2)$$

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, ..., a_n\}$
output: max of $\{a_1, a_2, ..., a_n\}$

```
max ← a_1
for i = 2 to n
    if max < a_i
        max ← a_i
return max
```

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, ..., a_n\}$
output: max of $\{a_1, a_2, ..., a_n\}$

```
max ← a_1
for i = 2 to n
     if max < a_i
        max ← a_i
return max
```

- T(n) : the number of operations the algorithm performs

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, ..., a_n\}$
output: max of $\{a_1, a_2, ..., a_n\}$

max ← a_1
for i = 2 to n
    if max < a_i
        max ← a_i
return max

2 op

- T(n) : the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, ..., a_n\}$
output: max of $\{a_1, a_2, ..., a_n\}$

max ← a_1
for i = 2 to n
    if max < a_i
        max ← a_i
return max

2 op

- T(n) : the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop
- loop's variable increases from 2 to n (use sum formula)

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, ..., a_n\}$
output: max of $\{a_1, a_2, ..., a_n\}$

max ← a_1 —— 1 op
for i = 2 to n
    if max < a_i
        max ← a_i —— 2 op
return max

- T(n) : the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop
- loop's variable increases from 2 to n (use sum formula)

$$C(n) = \sum_{i=2}^{n} 2 + 1$$

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, \ldots, a_n\}$
output: max of $\{a_1, a_2, \ldots, a_n\}$

```
max ← a_1                  ———— 1 op
for i = 2 to n
    if max < a_i           \
        max ← a_i          ——— 2 op
return max
```

- $\mathrm{T}(n)$: the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop
- loop's variable increases from 2 to n (use sum formula)

$$C(n) = \sum_{i=2}^{n} 2 + 1 = \sum_{i=1}^{n-1} 2 + 1$$

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, ..., a_n\}$
output: max of $\{a_1, a_2, ..., a_n\}$

max ← a_1 —— 1 op
for i = 2 to n
    if max < a_i
        max ← a_i —— 2 op
return max

- T(n) : the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop
- loop's variable increases from 2 to n (use sum formula)

$$C(n) = \sum_{i=2}^{n} 2 + 1 = \sum_{i=1}^{n-1} 2 + 1 = 2.(n-1) + 1 \in O(n)$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, ..., a_n\}$
output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n – 1
    for j = i+1 to n
        if a_i = a_j
        return false
return true
```

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \ldots, a_n\}$
output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n – 1
    for j = i+1 to n
        if a_i = a_j                1 op
        return false
return true
```

- the algorithm performs 1 operation on each execution of the innermost loop

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, ..., a_n\}$
output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n – 1
    for j = i+1 to n
        if a_i = a_j
        return false                1 op
return true
```

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to n – 1 for the outer loop, and from i + 1 to n for the innermost loop

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, ..., a_n\}$
output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n – 1
    for j = i+1 to n
        if a_i = a_j                 1 op
        return false
return true
```

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to n – 1 for the outer loop, and from i + 1 to n for the innermost loop

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \ldots, a_n\}$
output: return 'true' if all the elements are distinct; 'false' otherwise

for i = 1 to n – 1
  for j = i+1 to n
    if $a_i = a_j$
    return false — 1 op
return true

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to n – 1 for the outer loop, and from i + 1 to n for the innermost loop

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-1} [n - (i+1) + 1]$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \ldots, a_n\}$
output: return 'true' if all the elements are distinct; 'false' otherwise

for i = 1 to n – 1
    for j = i+1 to n
        if $a_i = a_j$
        return false — 1 op
return true

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to n – 1 for the outer loop, and from i + 1 to n for the innermost loop

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-1} [n - (i+1) + 1] = \sum_{i=1}^{n-1} (n - i)$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \ldots, a_n\}$
output: return 'true' if all the elements are distinct; 'false' otherwise

for i = 1 to n – 1
  for j = i+1 to n
    if $a_i = a_j$ — 1 op
    return false
return true

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to n – 1 for the outer loop, and from i + 1 to n for the innermost loop

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-1} [n - (i + 1) + 1] = \sum_{i=1}^{n-1} (n - i)$$

$$= \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \ldots, a_n\}$
output: return 'true' if all the elements are distinct; 'false' otherwise

for i = 1 to n – 1
    for j = i+1 to n
        if $a_i = a_j$
        return false   1 op
return true

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to n – 1 for the outer loop, and from i + 1 to n for the innermost loop

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-1} [n - (i+1) + 1] = \sum_{i=1}^{n-1} (n - i)$$

$$= \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i = n(n-1) - \frac{n(n-1)}{2}$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \ldots, a_n\}$
output: return 'true' if all the elements are distinct; 'false' otherwise

for i = 1 to n – 1
  for j = i+1 to n
    if $a_i = a_j$ — 1 op
    return false
return true

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to n – 1 for the outer loop, and from i + 1 to n for the innermost loop

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-1} [n - (i+1) + 1] = \sum_{i=1}^{n-1} (n - i)$$

$$= \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i = n(n-1) - \frac{n(n-1)}{2} = \frac{1}{2} n(n-1) \in O(n^2)$$

Worst-Case Analysis

mxn matrix A nxk matrix B   mxk matrix C

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} \;\; a_{i2} & \ldots & a_{in} \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} . \begin{pmatrix} b_{11} & b_{1j} & b_{1k} \\ \vdots & b_{2j} & \vdots \\ & \vdots & \\ b_{n1} & b_{nj} & b_{nk} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mk} \end{pmatrix}$$

$$c_{ij} = a_{i1}.b_{1j} + a_{i2}.b_{2j} + \cdots + a_{in}.b_{nj}$$

UniqueElements(list)

input : two nxn matrices A,B
output: C = A.B

for i = 1 to n
    for j = 1 to n
        $C[i,j] \leftarrow 0$
        for k = 1 to n
            $C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$
return C

Worst-Case Analysis

UniqueElements(list)

input : two nxn matrices A,B
output: A.B

```
for i = 1 to n
    for j = 1 to n
        C[i,j] ← 0
        for k = 1 to n
            C[i,j] ← C[i,j] + A[i,k] * B[k,j]
return C
```

Worst-Case Analysis

UniqueElements(list)

input : two nxn matrices A,B
output: A.B

for i = 1 to n
  for j = 1 to n
    $C[i,j] \leftarrow 0$ — 1 op
    for k = 1 to n
      $C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$ — 2 op
return C

Worst-Case Analysis

UniqueElements(list)

input : two nxn matrices A,B
output: A.B

```
for i = 1 to n
    for j = 1 to n
        C[i,j] ← 0                              1 op
        for k = 1 to n
            C[i,j] ← C[i,j] + A[i,k] * B[k,j]   2 op
return C
```

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} (1 + \ \dots)$$

Worst-Case Analysis

UniqueElements(list)

input : two nxn matrices A,B
output: A.B

for i = 1 to n
    for j = 1 to n
        $C[i,j] \leftarrow 0$ — 1 op
        for k = 1 to n
            $C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$ — 2 op
return C

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(1 + \sum_{k=1}^{n} 2\right)$$

Worst-Case Analysis

UniqueElements(list)

input : two nxn matrices A,B
output: A.B

for i = 1 to n
  for j = 1 to n
    $C[i,j] \leftarrow 0$ —— 1 op
    for k = 1 to n
      $C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$ —— 2 op
return C

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(1 + \sum_{k=1}^{n} 2\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} (2n+1)$$

Worst-Case Analysis

UniqueElements(list)

input : two nxn matrices A,B
output: A.B

for i = 1 to n
    for j = 1 to n
        $C[i,j] \leftarrow 0$ — 1 op
        for k = 1 to n
            $C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$ — 2 op
return C

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(1 + \sum_{k=1}^{n} 2\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} (2n+1) = \sum_{i=1}^{n} n(2n+1)$$

Worst-Case Analysis

UniqueElements(list)

input : two nxn matrices A,B
output: A.B

for i = 1 to n
    for j = 1 to n
        $C[i,j] \leftarrow 0$ — 1 op
        for k = 1 to n
            $C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$ — 2 op
return C

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(1 + \sum_{k=1}^{n} 2\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} (2n+1) = \sum_{i=1}^{n} n(2n+1)$$

$$= n.n.(2n+1)$$

Worst-Case Analysis

UniqueElements(list)

input : two nxn matrices A,B
output: A.B

```
for i = 1 to n
    for j = 1 to n
        C[i,j] ← 0                                    1 op
        for k = 1 to n
            C[i,j] ← C[i,j] + A[i,k] * B[k,j]         2 op
return C
```

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(1 + \sum_{k=1}^{n} 2\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} (2n+1) = \sum_{i=1}^{n} n(2n+1)$$

$$= n.n.(2n+1) = 2n^3 + n^2 \in O(n^3)$$

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```

can be ignored

since constant number of operations can be counted as 1 operation, it can be ignored

$$T(n) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+\sum_{k=1}^{n}2\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}(2n+1) = \sum_{i=1}^{n}n(2n+1)$$

$$= n.n.(2n+1) = 2n^3 + n^2 \in O(n^3)$$

$$T(n) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}1 = n^3 \in O(n^3)$$

Worst-Case Analysis

BinarySearch(X,i,j;x)

input : $\{X = \{a_1, a_2, \ldots, a_n\}; x\}$
output: 'yes' if $x \in X$, 'no' otherwise

```
if i = j
    if  x = a_i
        return 'yes'
    else
        return 'no'
elseif x < a_⌊(i+j)/2⌋
        BinarySearch(X,i, ⌊(i+j)/2⌋-1;x)
else BinarySearch(X,⌊(i+j)/2⌋+1,j;x)
```

MergeSort(X,i,j)

input : $\{X = \{a_1, a_2, \ldots, a_n\}\}$
output: sorted sequence of elements in X

```
if i < j
    k = ⌊(i+j)/2⌋
    MergeSort(X,i,k)
    MergeSort(X,k+1,j)
    Merge(A,i,k,j)
```

Worst-Case Analysis

BinarySearch(X,i,j;x)

input : $\{X = \{a_1, a_2, ..., a_n\}; x\}$
output: 'yes' if $x \in X$, 'no' otherwise

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- Let $T(n)$ be the number of operations BinarySearch performs on an input of size n, then

$$T(n) = T(n/2) + 1, \text{ and } T(n) = 1 \text{ for } n \leq 2$$

Worst-Case Analysis

BinarySearch(X,i,j;x)

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MergeSort(X,i,j)

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$$T(n) = T(n/2) + 1, \text{ and } T(n) = 1 \text{ for } n \leq 2$$

- Let $T(n)$ be the number of operations MergeSort performs on an input of size n, then

$$T(n) = 2T(n/2) + \Theta(n), \text{ and } T(n) = 1 \text{ for } n = 1$$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1 & , \quad if \ \mathrm{n} \leq 2 \\ 2T(n/2) + n & , \quad if \ n > 2 \end{cases}$$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1 & , \quad if\ \mathrm{n} \leq 2 \\ 2T(n/2) + n & , \quad if\ n > 2 \end{cases}$$

- $T(n) = 2[2T(n/4) + n/2] + n$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1 & , \quad if\ \text{n} \leq 2 \\ 2T(n/2) + n & , \quad if\ n > 2 \end{cases}$$

- $T(n) = 2[2T(n/4) + n/2] + n = 2^2 T(n/2^2) + 2n$

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Worst-Case Analysis

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Worst-Case Analysis

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  $T(n) = 2^iT(n/2^i) + i.n$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

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- Base Case : $n/2^i = 2$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

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  $T(n) = 2^i T(n/2^i) + i.n$

- Base Case : $n/2^i = 2 \rightarrow n = 2^{i+1}$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

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  $T(n) = 2^iT(n/2^i) + i.n$

- Base Case : $n/2^i = 2 \rightarrow n = 2^{i+1} \rightarrow i = \log n - 1$

Worst-Case Analysis

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- Base Case : $n/2^i = 2 \rightarrow n = 2^{i+1} \rightarrow i = \log n - 1$

- Thus, $T(n) = 2^iT\left(n/2^i\right) + i.n = 2^{\log n - 1}T(2) + (\log n - 1)n$

Worst-Case Analysis

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  $$a^{log_b n} = n^{log_b a}$$

  $T(n) = 2^iT(n/2^i) + i.n$

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- Thus, $T(n) = 2^iT\left(n/2^i\right) + i.n = 2^{\log n - 1}T(2) + (\log n - 1)n$

Worst-Case Analysis

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Worst-Case Analysis

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$$T(n) = \begin{cases} 1 & , \quad if \ \mathrm{n} \le 2 \\ 2T(n/2) + n & , \quad if \ n > 2 \end{cases}$$

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  $T(n) = (n/2) + n\log n - n$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

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- Thus, $T(n) = 2^iT\left(n/2^i\right) + i.n = 2^{\log n - 1}T(2) + (\log n - 1)n$

  $T(n) = (n/2) + n\log n - n = n\log n - (n/2)$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1 & , \quad if\ \text{n} \leq 2 \\ 2T(n/2) + n & , \quad if\ n > 2 \end{cases}$$

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  $T(n) = (n/2) + n\log n\ - n = n\log n\ - (n/2) = O(n\log n)\ \big(= \Theta(n\log n)\big)$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1, & if\ \mathrm{n} \leq 2 \\ 5T(n/2) + n^2, & if\ n > 2 \end{cases}$$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1 \quad , & if \ \mathrm{n} \leq 2 \\ 5T(n/2) + n^2, & if \ n > 2 \end{cases}$$

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Worst-Case Analysis

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Worst-Case Analysis

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Worst-Case Analysis

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  $T(n) = 5^iT(n/2^i) + \left[(5/4)^{i-1} + \cdots + (5/4) + 1\right]n^2$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1, & if\ \text{n} \le 2 \\ 5T(n/2) + n^2, & if\ n > 2 \end{cases}$$

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Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1 \quad , & if \ \mathrm{n} \leq 2 \\ 5T(n/2) + n^2, & if \ n > 2 \end{cases}$$

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  $T(n) = 5^3T(n/2^3) + (5/4)^2n^2 + (5/4)n^2 + n^2$
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- Base Case : $n/2^i = 2$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1, & if\ \mathrm{n} \leq 2 \\ 5T(n/2) + n^2, & if\ n > 2 \end{cases}$$

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- Base Case : $n/2^i = 2 \rightarrow n = 2^{i+1}$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

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- Base Case : $n/2^i = 2 \rightarrow n = 2^{i+1} \rightarrow i = \log n \ - 1$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1, & if \ \mathrm{n} \leq 2 \\ 5T(n/2) + n^2, & if \ n > 2 \end{cases}$$

- $T(n) = 5[5T(n/4) + (n/2)^2] + n^2 = 5^2T(n/2^2) + (5/4)n^2 + n^2$
  $T(n) = 5^2[5T(n/2^3) + (n/2^2)^2] + (5/4)n^2 + n^2$
  $T(n) = 5^3T(n/2^3) + (5/4)^2n^2 + (5/4)n^2 + n^2$
  $T(n) = 5^iT(n/2^i) + \left[(5/4)^{i-1} + \cdots + (5/4) + 1\right]n^2 = 5^iT(n/2^i) + \frac{(5/4)^i-1}{(5/4)-1}n^2$
- Base Case : $n/2^i = 2 \rightarrow n = 2^{i+1} \rightarrow i = \log n \ - 1$
- Thus, $T(n) = 5^iT\left(n/2^i\right) + 4\left[(5/4)^i-1\right]n^2$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1, & if\ \mathrm{n} \leq 2 \\ 5T(n/2) + n^2, & if\ n > 2 \end{cases}$$

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- Base Case : $n/2^i = 2 \rightarrow n = 2^{i+1} \rightarrow i = \log n - 1$

- Thus, $T(n) = 5^iT\left(n/2^i\right) + 4\left[(5/4)^i - 1\right]n^2 = 5^{\log n - 1}T(2) + 4\left[(5/4)^{\log n - 1} - 1\right]n^2$

Worst-Case Analysis

Backward Substitution for Recurrence Relation

$$T(n) = \begin{cases} 1 \quad , & if \ \text{n} \leq 2 \\ 5T(n/2) + n^2, & if \ n > 2 \end{cases}$$

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Worst-Case Analysis

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  $T(n) = O\left(n^{\log 5}\right)$ $(= \Theta(n^{\log 5}))$

Worst-Case Analysis

Recursion Tree for Recurrence Relation

$$T(n) = \begin{cases} 1, & if \ \mathrm{n} \leq 1 \\ T(n/4) + T(n/2) + n^2, & if \ n > 1 \end{cases}$$

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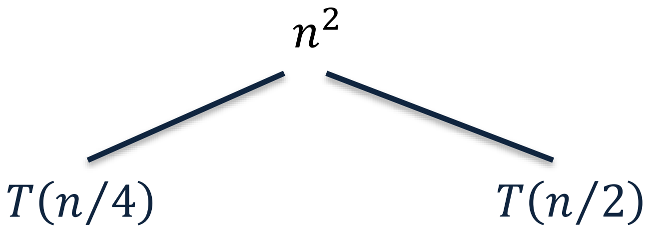

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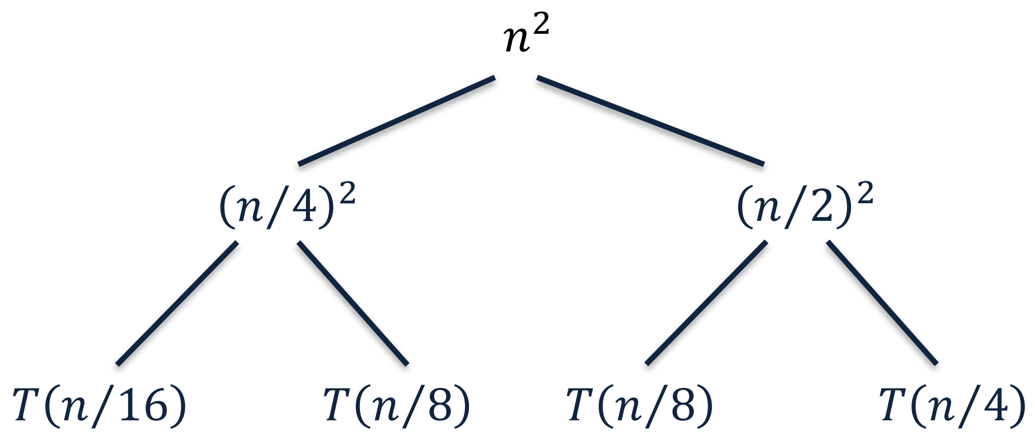

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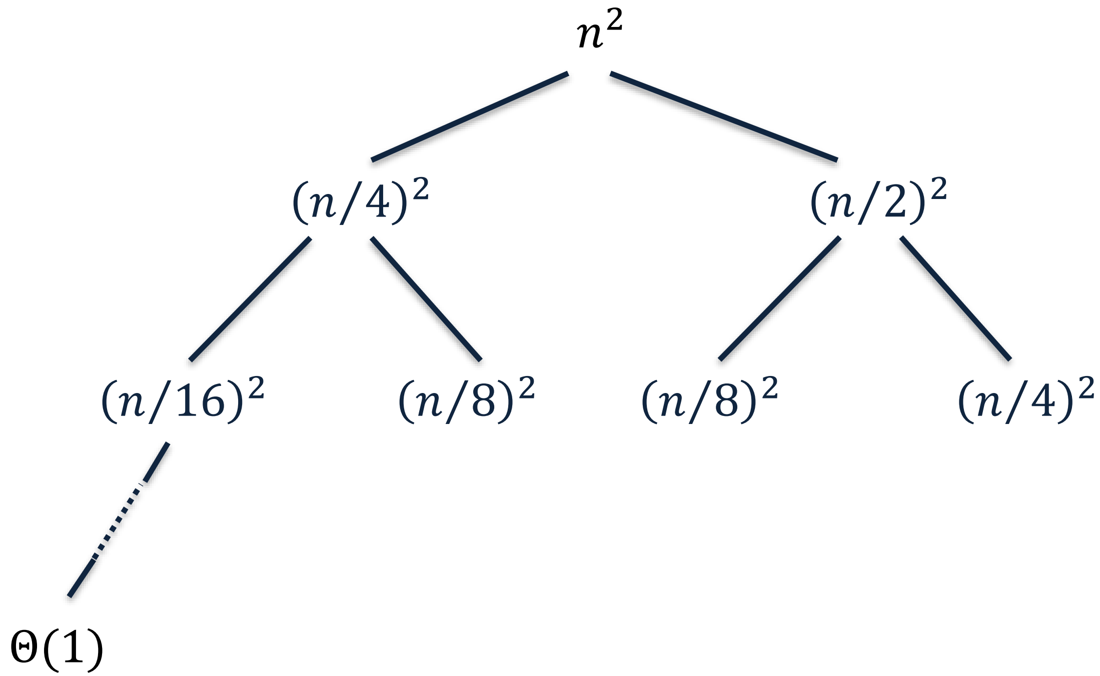

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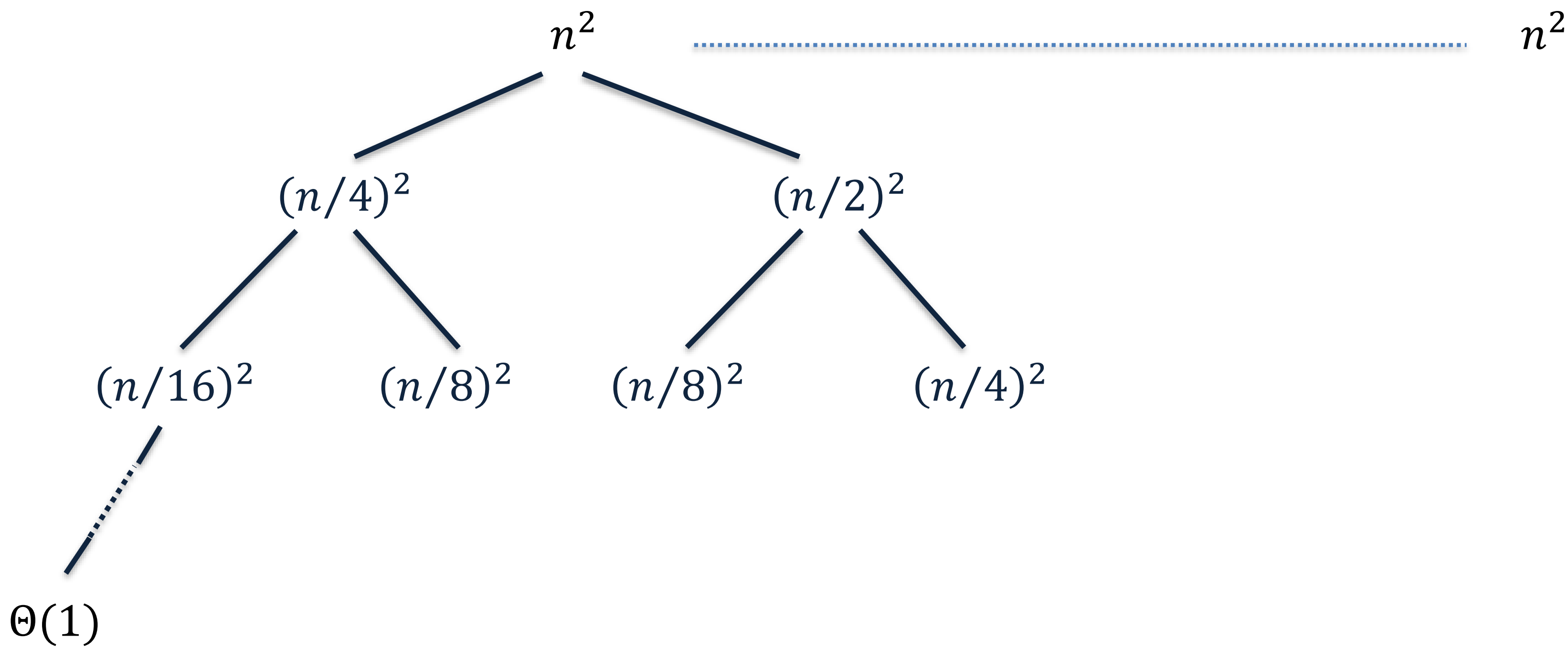

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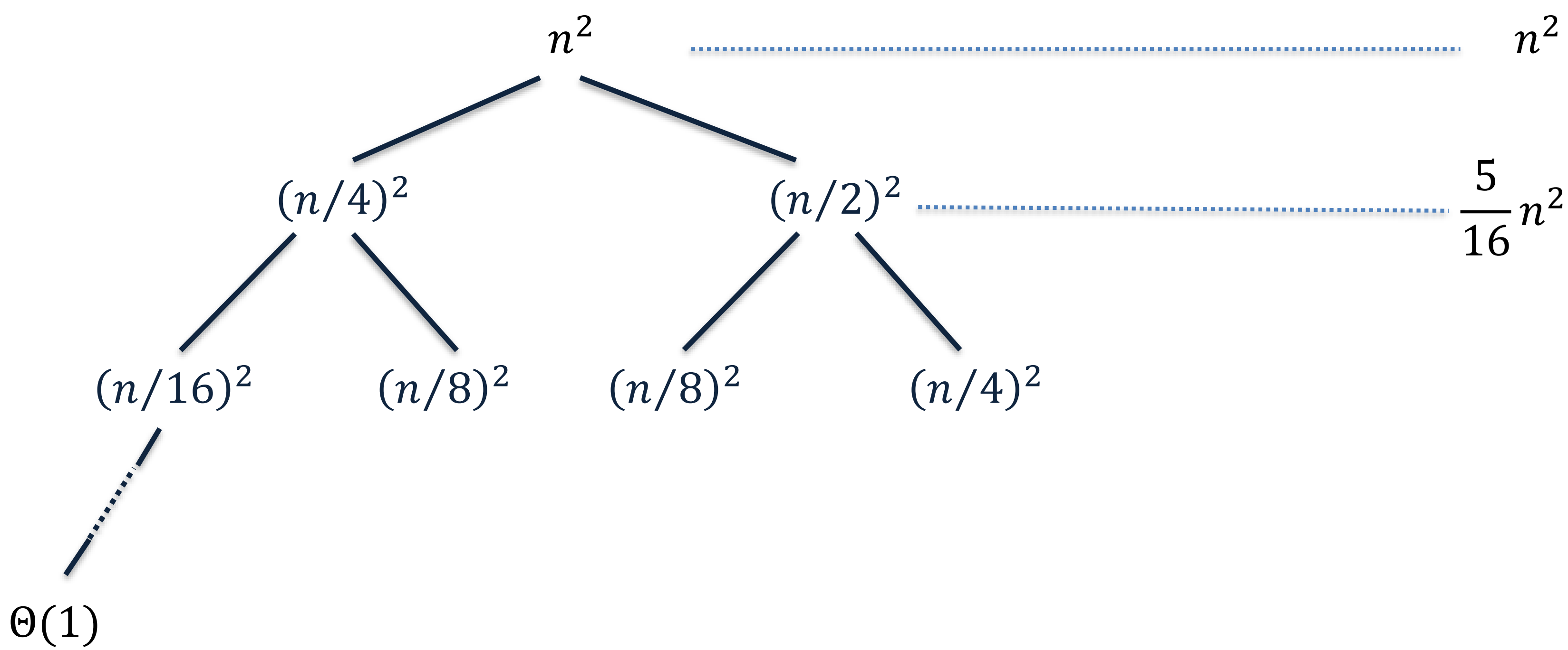

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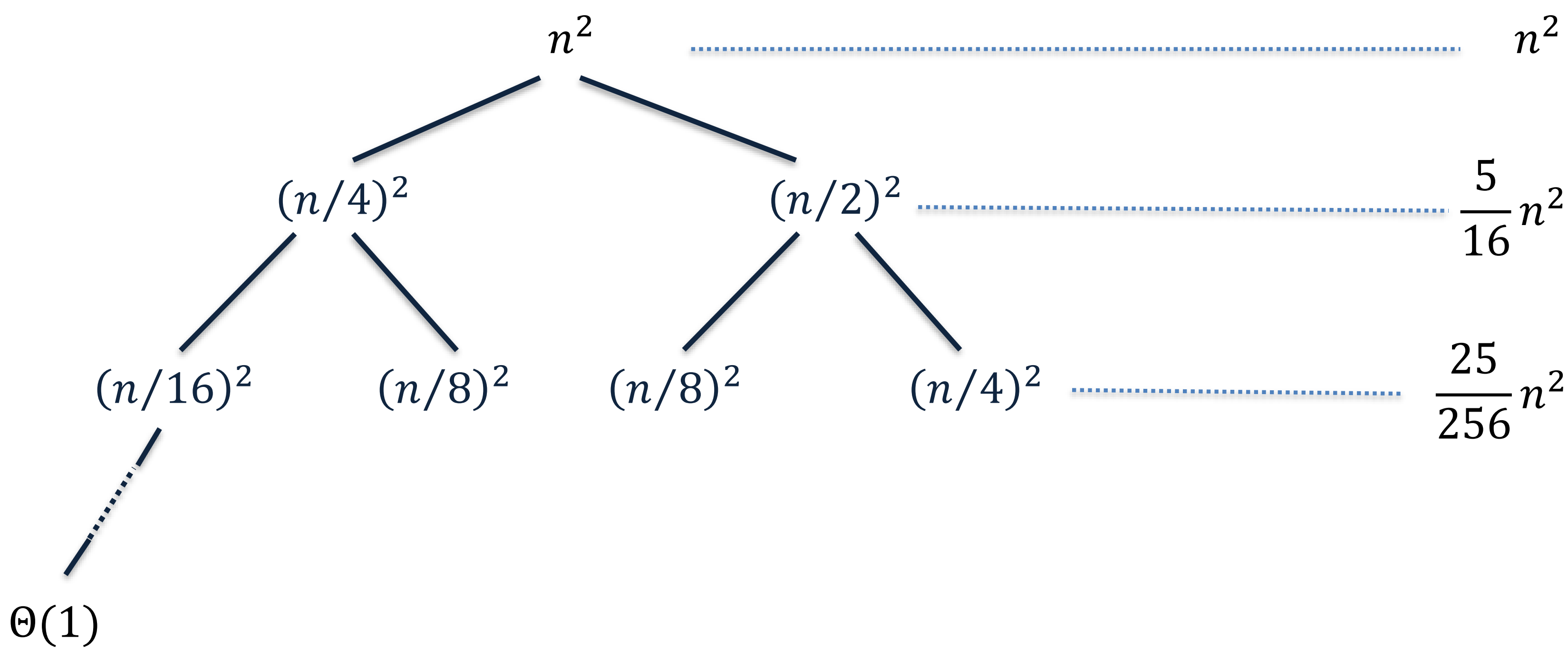

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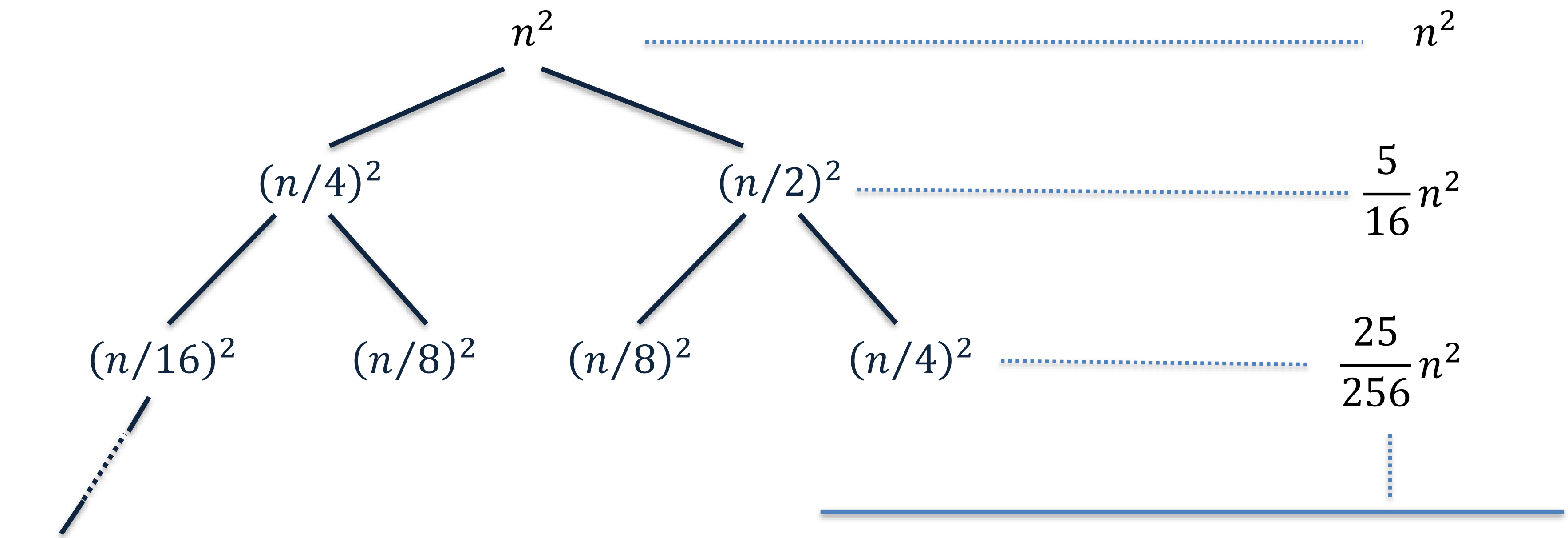

$$\text{total} = n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$

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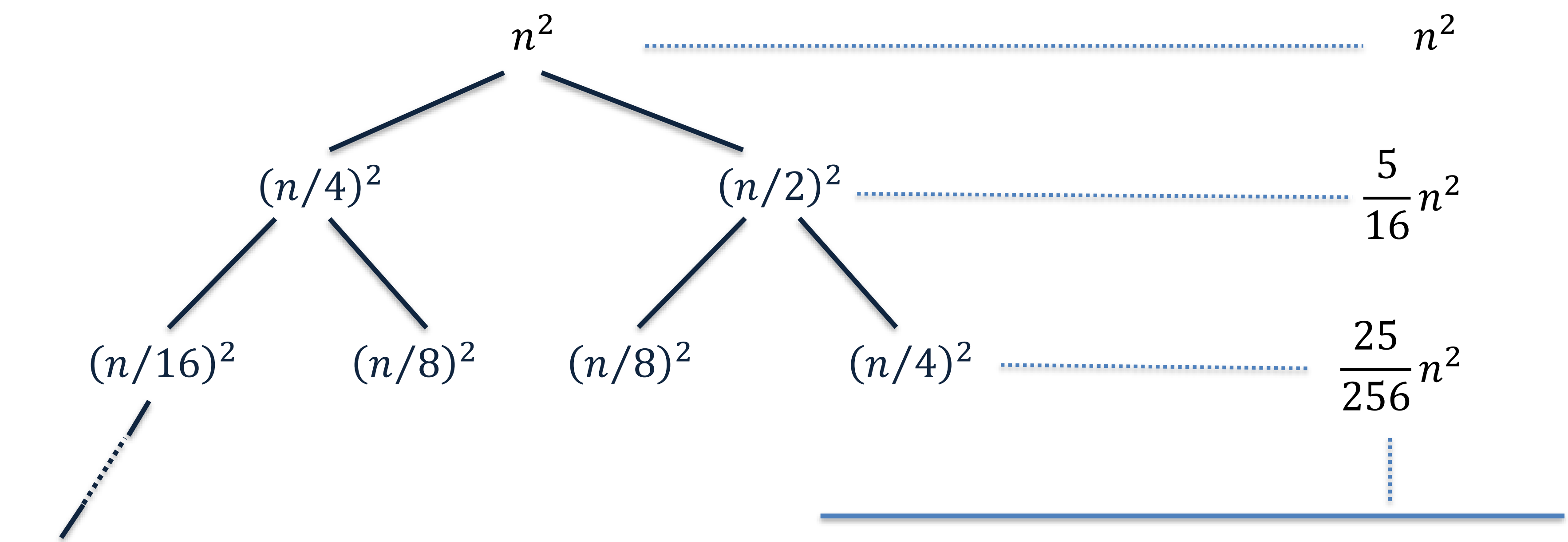

$\Theta(1)$

$$\text{total} = n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$

$$= \frac{16}{11}n^2$$

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \text{ if } |x| < 1$$

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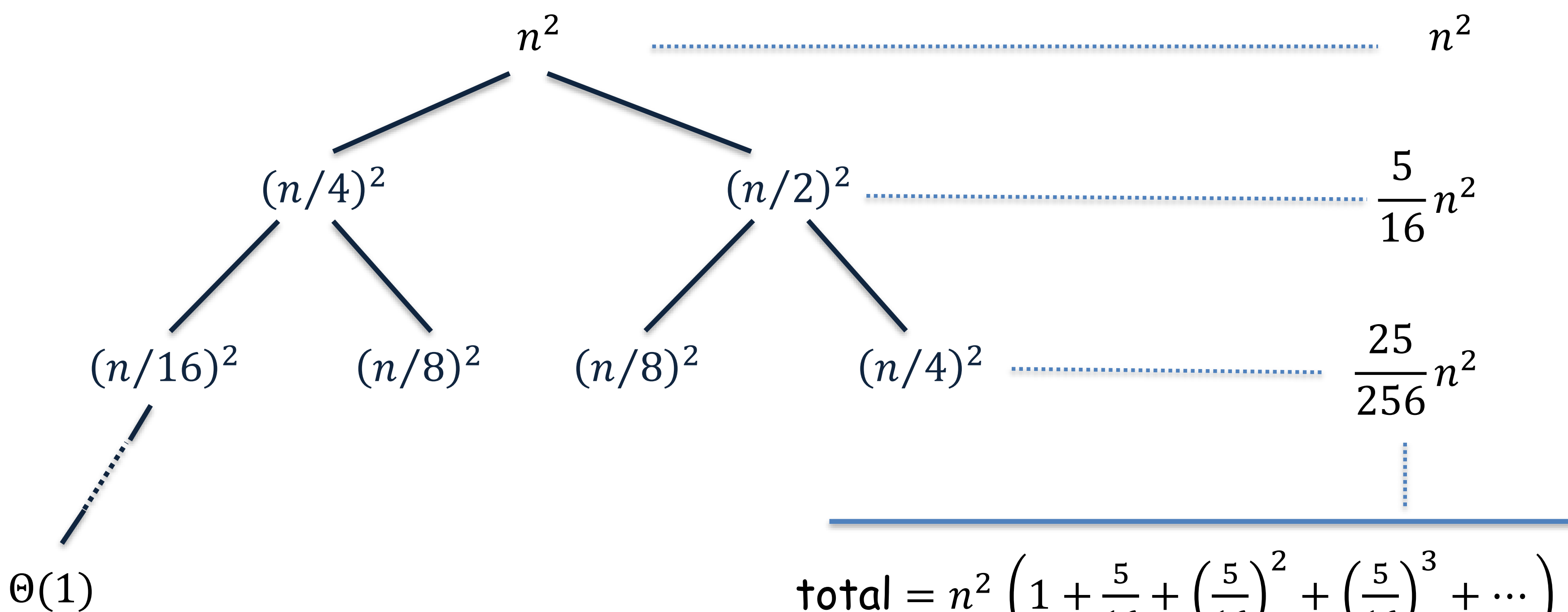

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Master Theorem

$$T(n) = \begin{cases} \Theta(1) & , \quad if \; \mathrm{n} \leq 1 \\ aT(n/b) + f(n) & , \quad \text{otherwise} \end{cases}$$

where $a \geq 1$, $b > 1$, and $f(n)$ is a non-negative integer function

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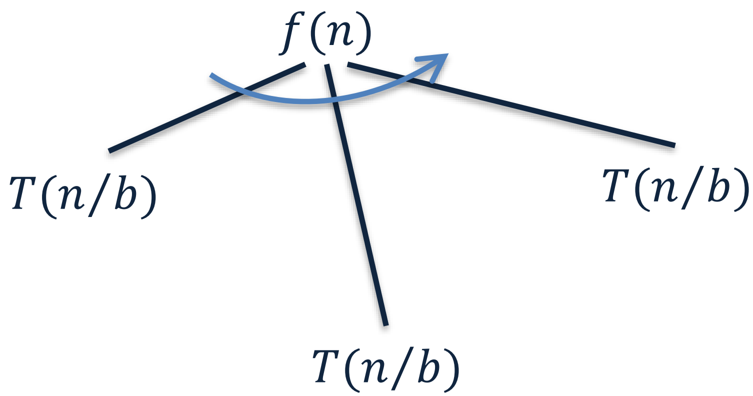

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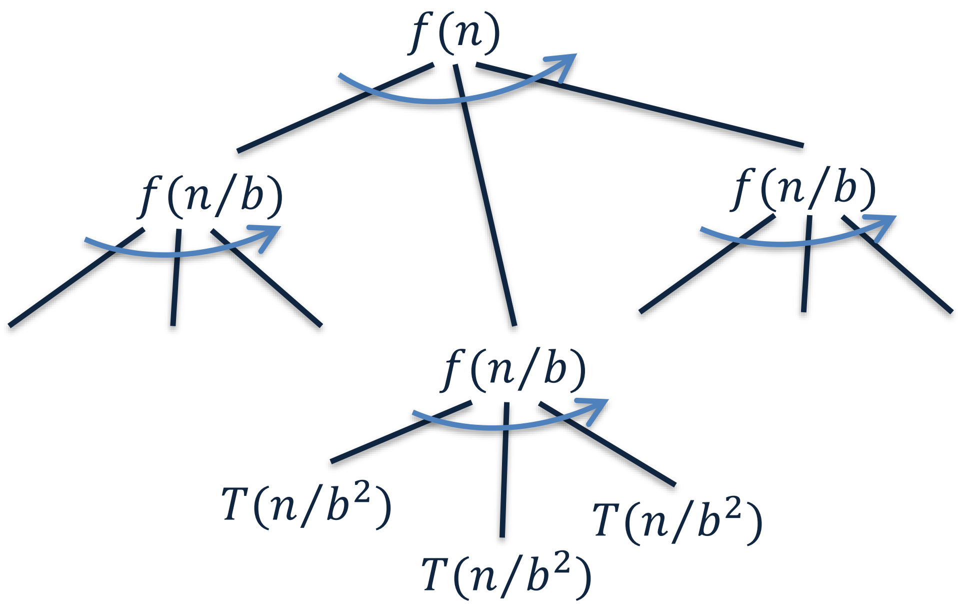

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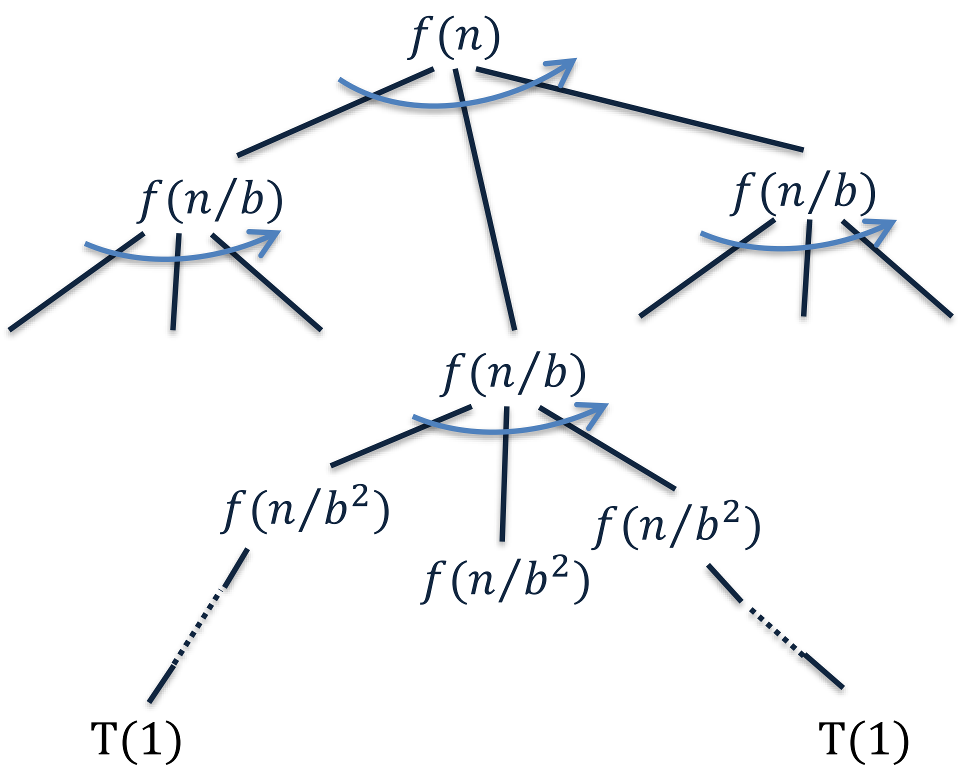

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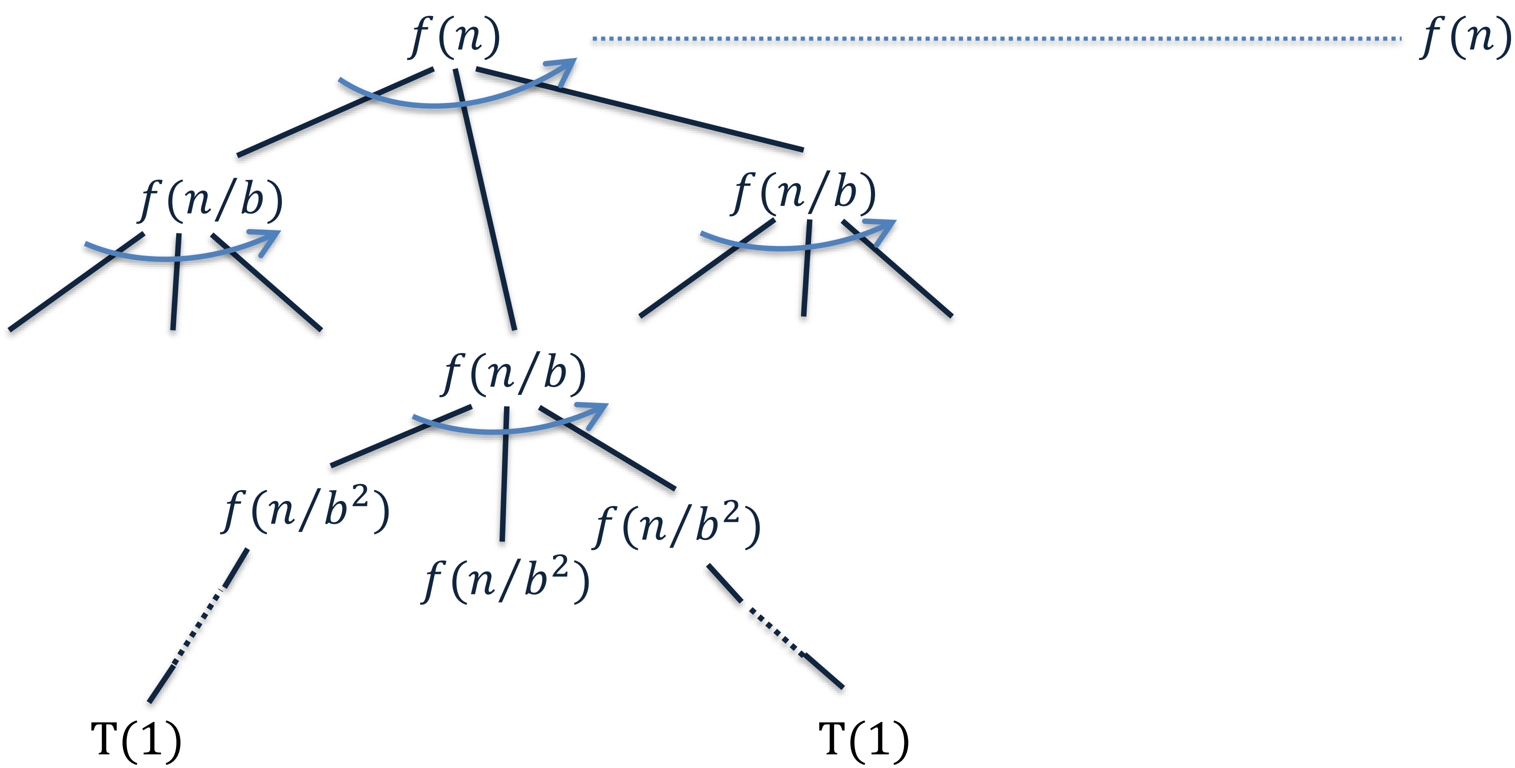

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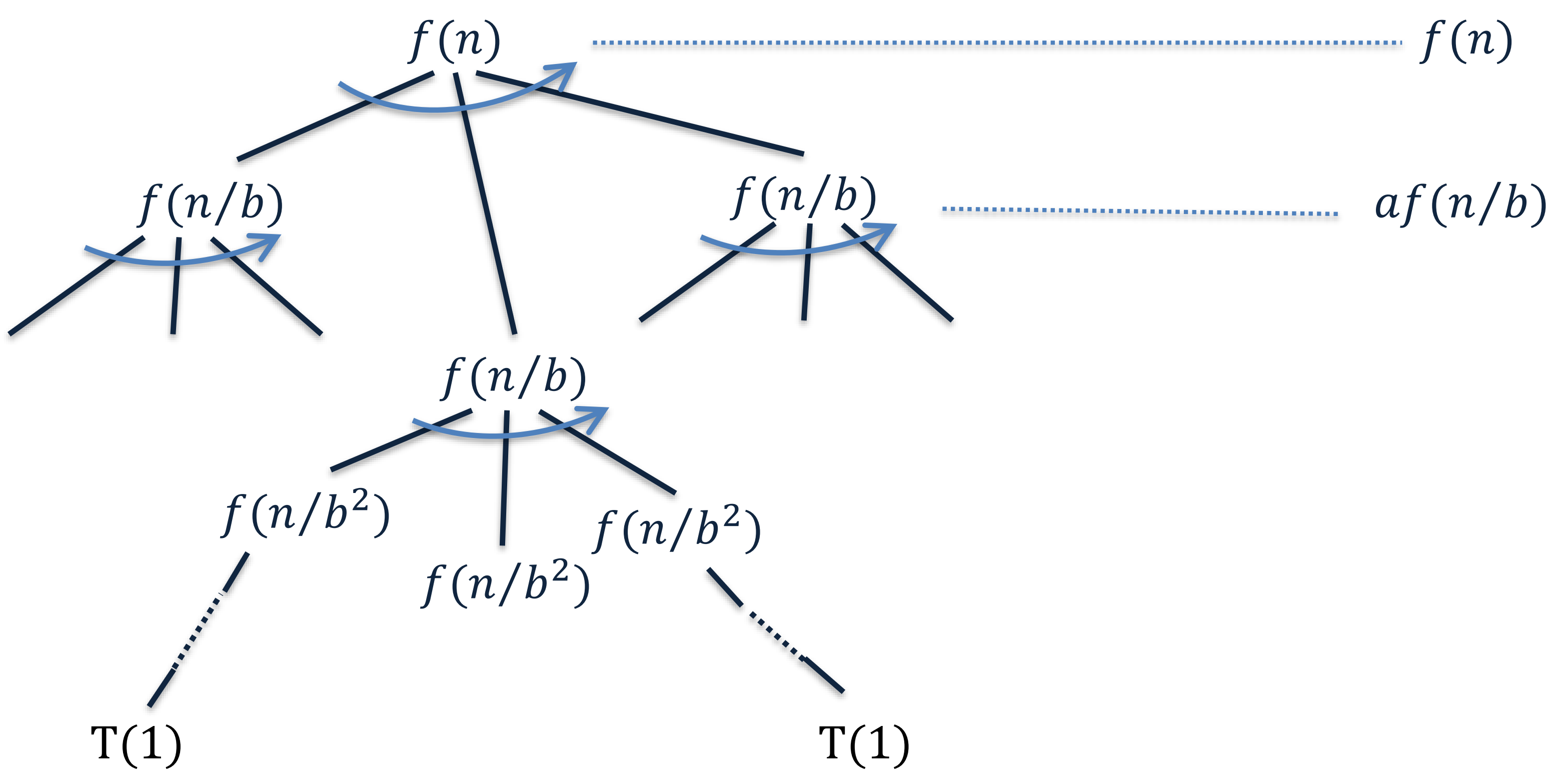

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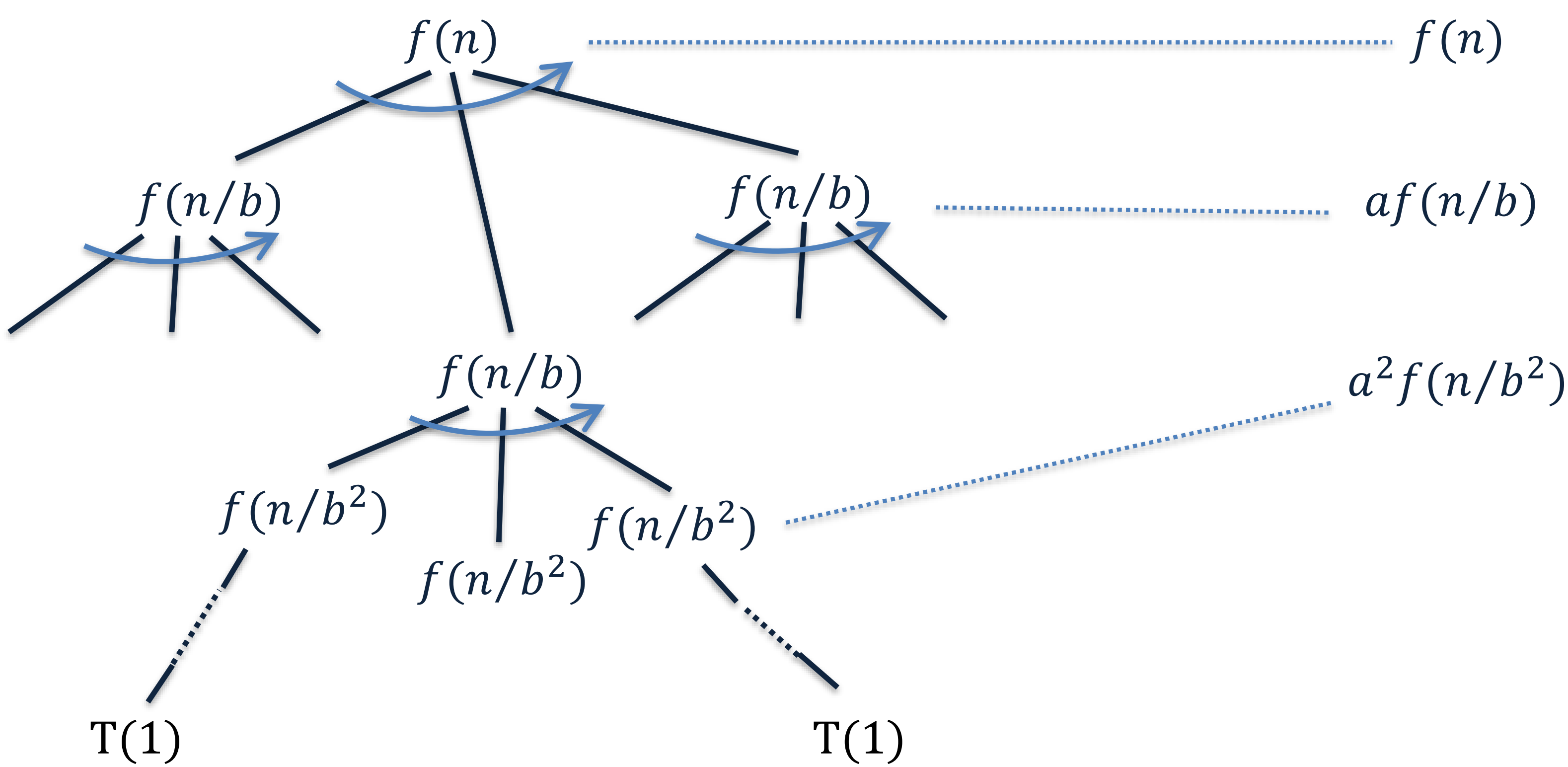

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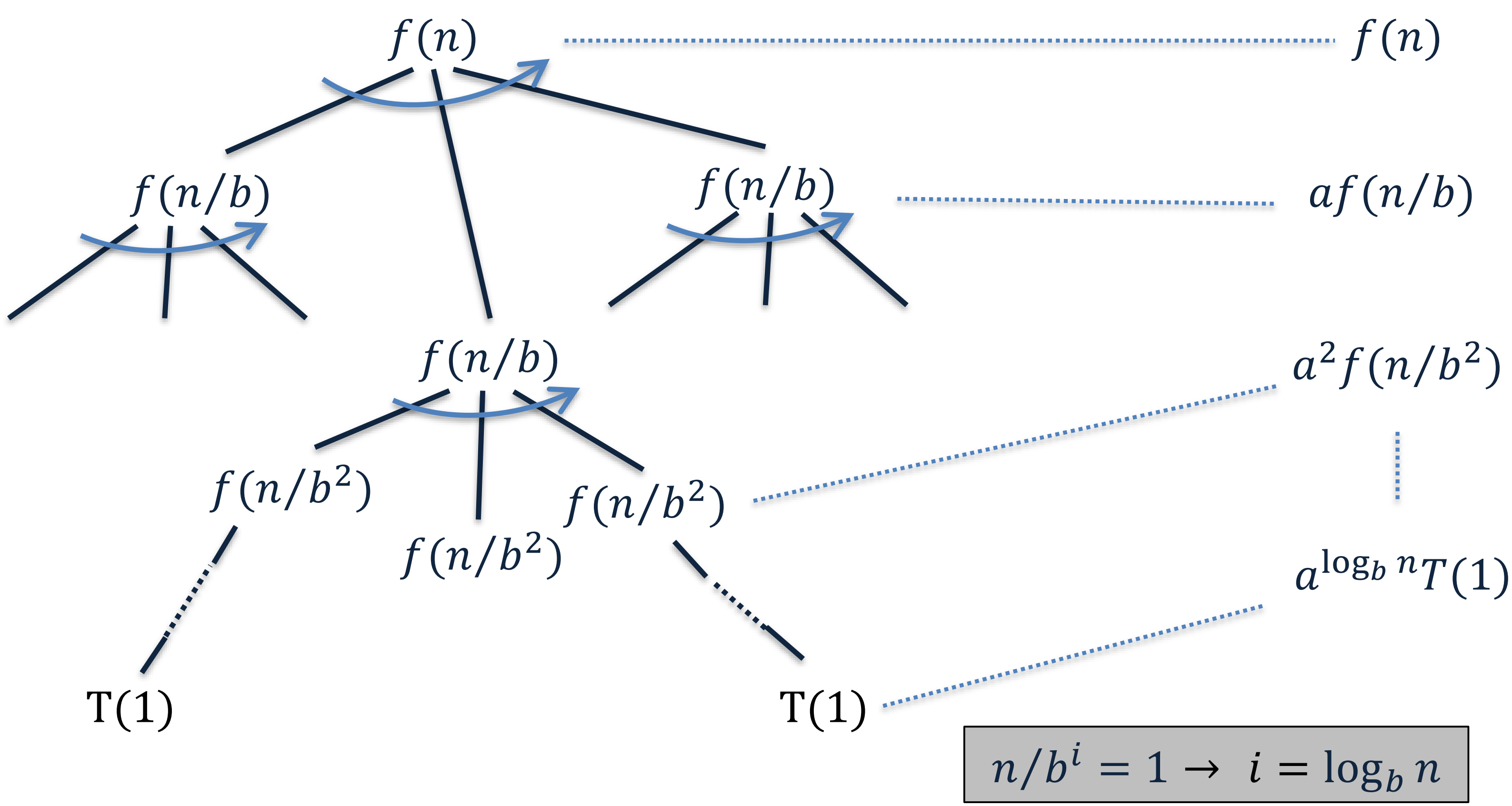

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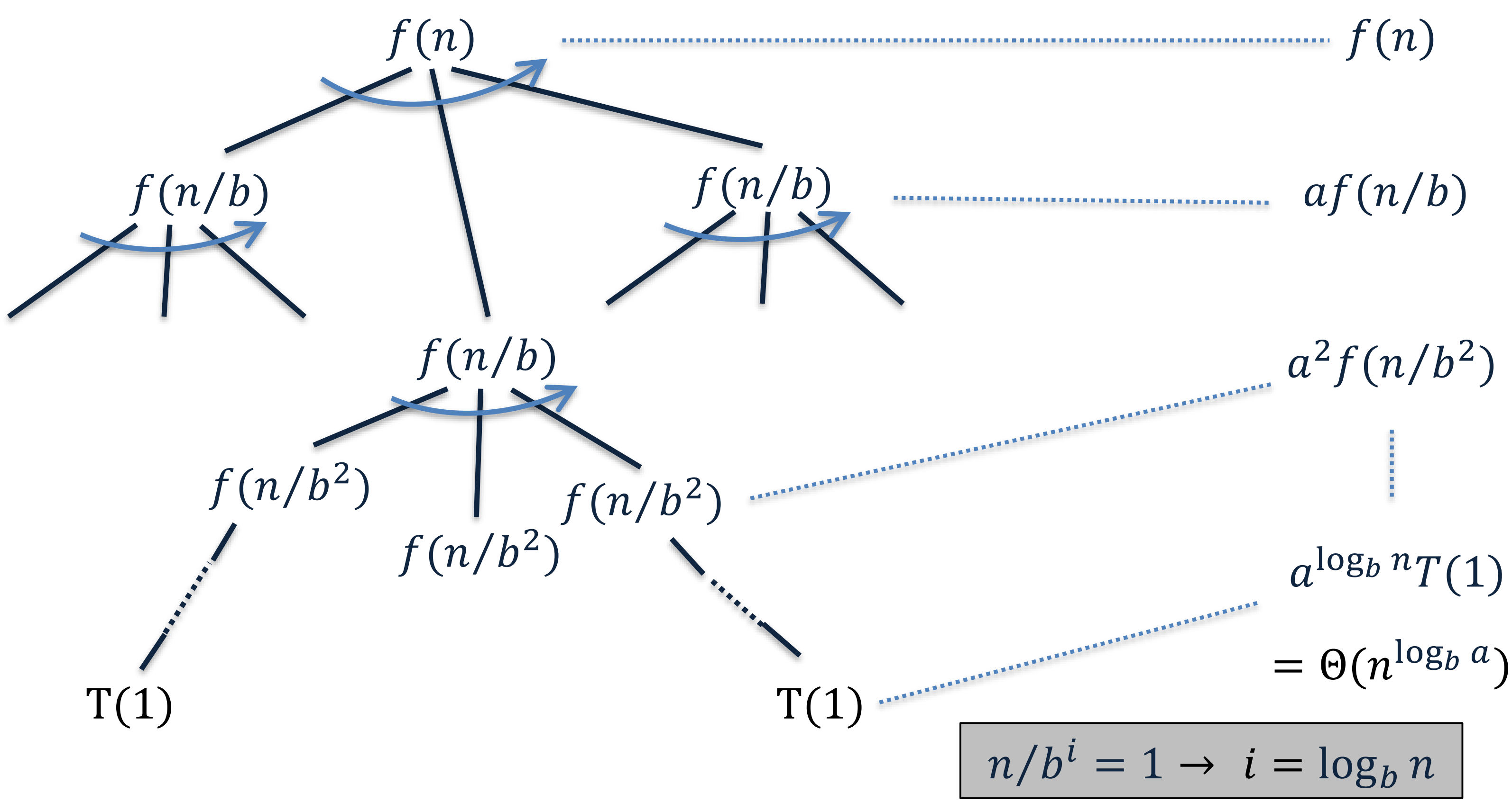

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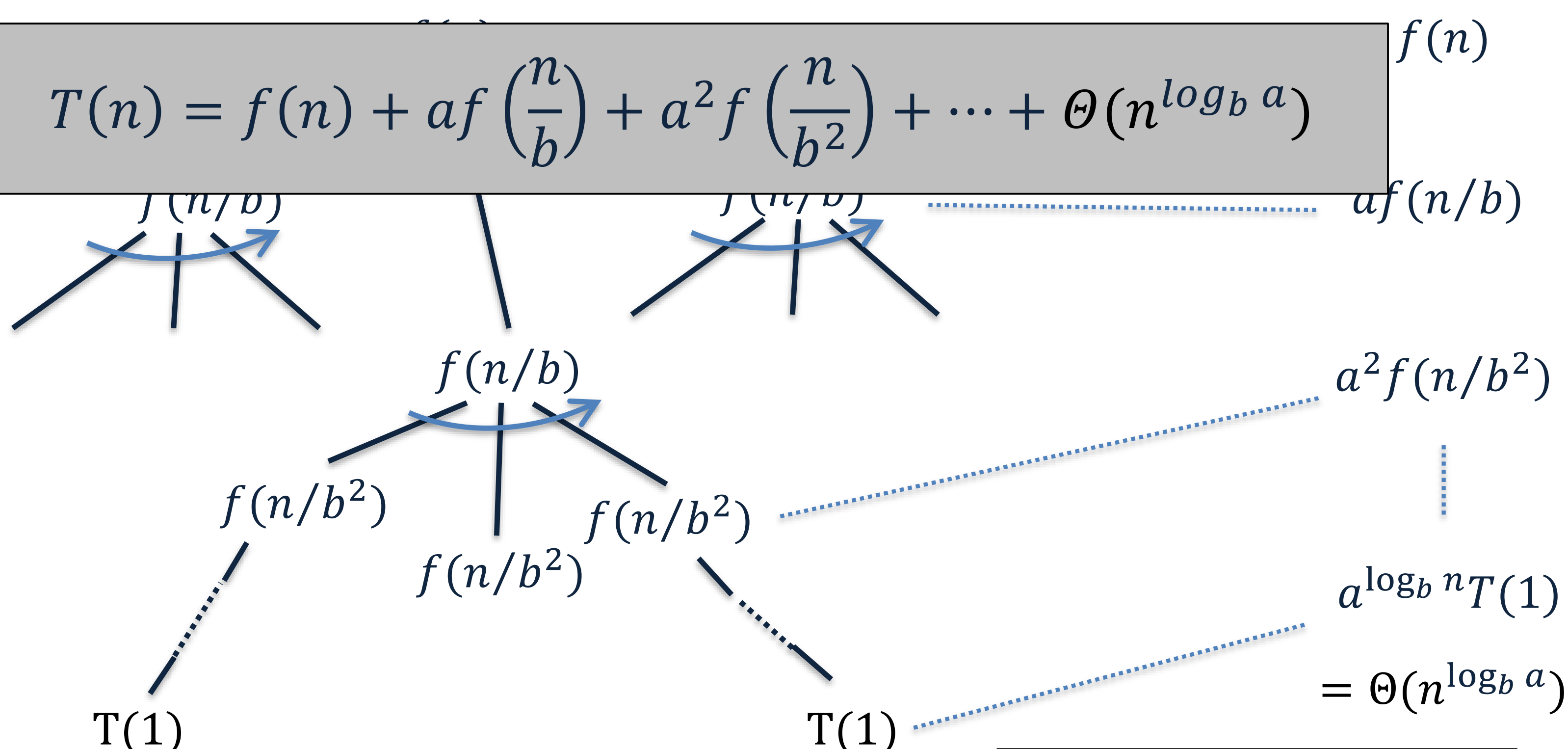

$n/b^i = 1 \rightarrow i = \log_b n$

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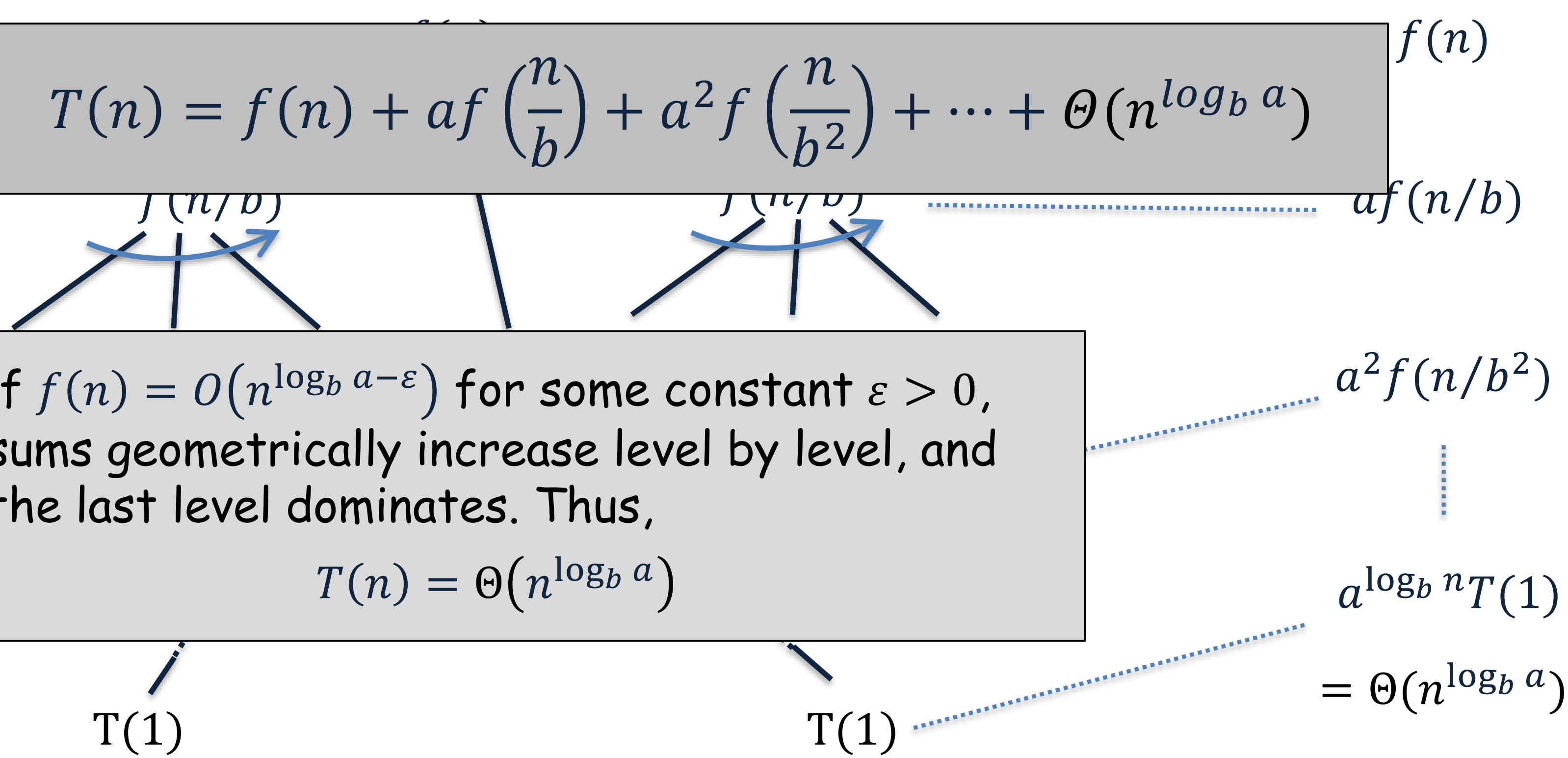


- if $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$ for some constant $\varepsilon > 0$, sums geometrically increase level by level, and the last level dominates. Thus,

$$T(n) = \Theta\left(n^{\log_b a}\right)$$

Worst-Case Analysis

Master Theorem

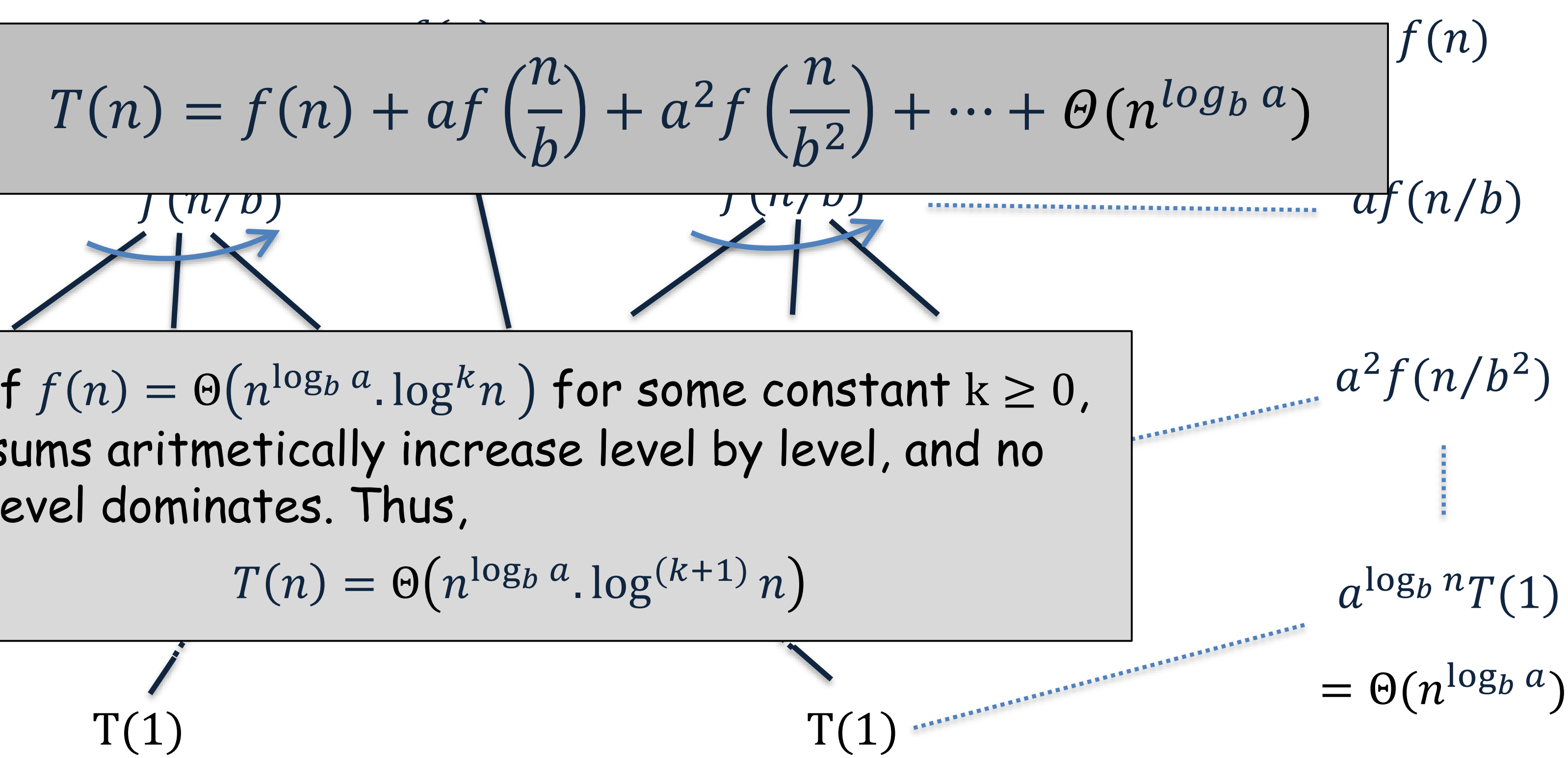
$$T(n) = \begin{cases} \Theta(1) \quad , & if\ n \le 1 \\ aT(n/b) + f(n) \ , & \text{otherwise} \end{cases}$$

where $a \ge 1$, $b > 1$, and $f(n)$ is a non-negative integer function

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \cdots + \Theta\left(n^{\log_b a}\right)$$

- if $f(n) = \Theta\left(n^{\log_b a} . \log^k n\right)$ for some constant $k \ge 0$, sums aritmetically increase level by level, and no level dominates. Thus,

$$T(n) = \Theta\left(n^{\log_b a} . \log^{(k+1)} n\right)$$

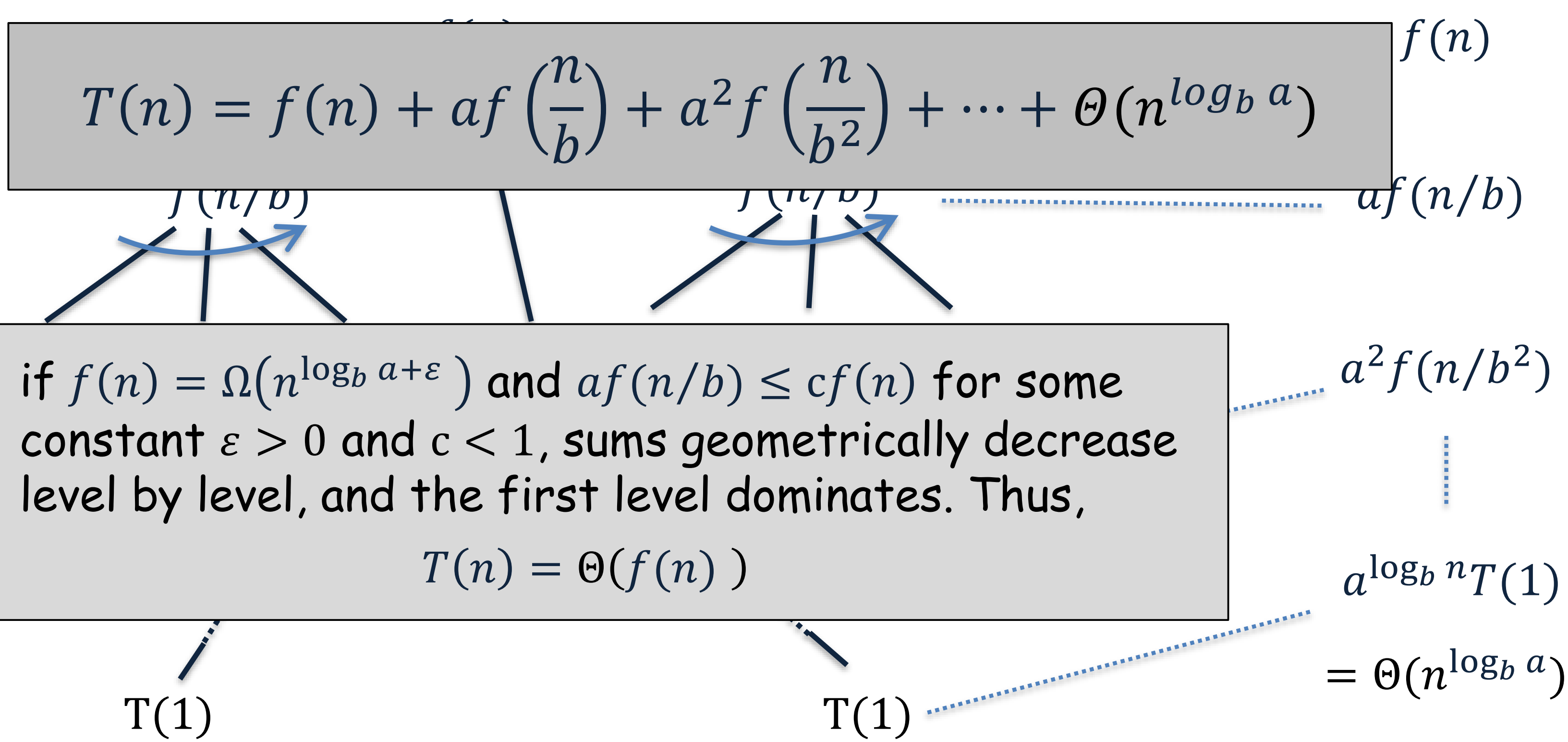


Worst-Case Analysis

Master Theorem

$$T(n) = \begin{cases} \Theta(1) & , \quad if\ n \le 1 \\ aT(n/b) + f(n) & , \quad \text{otherwise} \end{cases}$$

where $a \ge 1$, $b > 1$, and $f(n)$ is a non-negative integer function



$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \cdots + \Theta(n^{\log_b a})$$

- if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and $af(n/b) \le cf(n)$ for some constant $\varepsilon > 0$ and $c < 1$, sums geometrically decrease level by level, and the first level dominates. Thus,

$$T(n) = \Theta(f(n))$$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^2$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^2$
  - $a = 4,\ b = 2,\ f(n) = n^2$, thus $n^{\log_b a} = n^2$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^2$
  - $a = 4,\ b = 2,\ f(n) = n^2$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Theta(n^2 . \log^0 n)$ for $k = 0$ and it's the second case

Worst-Case Analysis

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- $T(n) = 4T(n/2) + n^2$
  - $a = 4,\ b = 2,\ f(n) = n^2$, thus $n^{\log_b a} = n^2$
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  - thus, $T(n) = \Theta(n^2 . \log n)$

Worst-Case Analysis

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  - thus, $T(n) = \Theta(n^2 . \log n)$
- $T(n) = 3T(n/2) + n$

Worst-Case Analysis

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- $T(n) = 4T(n/2) + n^2$
  - $a = 4,\ b = 2,\ f(n) = n^2$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Theta(n^2 . \log^0 n)$ for $k = 0$ and it's the second case
  - thus, $T(n) = \Theta(n^2 . \log n)$
- $T(n) = 3T(n/2) + n$
  - $a = 3,\ b = 2,\ f(n) = n$, thus $n^{\log_b a} = n^{\log_2 3}$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^2$
  - $a = 4,\ b = 2,\ f(n) = n^2$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Theta(n^2.\log^0 n)$ for $k = 0$ and it's the second case
  - thus, $T(n) = \Theta(n^2.\log n)$
- $T(n) = 3T(n/2) + n$
  - $a = 3,\ b = 2,\ f(n) = n$, thus $n^{\log_b a} = n^{\log_2 3}$
  - since $f(n) = O\left(n^{\log_2 3 - \varepsilon}\right)$ for $\varepsilon = \log_2(3/2)$ and it's the first case

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^2$
  - $a = 4,\ b = 2,\ f(n) = n^2$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Theta(n^2 . \log^0 n)$ for $k = 0$ and it's the second case
  - thus, $T(n) = \Theta(n^2 . \log n)$
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Worst-Case Analysis

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  - $a = 4,\ b = 2,\ f(n) = n^2$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Theta(n^2.\log^0 n\ )$ for $k = 0$ and it's the second case
  - thus, $T(n) = \Theta(n^2.\log n)$
- $T(n) = 3T(n/2) + n$
  - $a = 3,\ b = 2,\ f(n) = n$, thus $n^{\log_b a} = n^{\log_2 3}$
  - since $f(n) = O\left(n^{\log_2 3 - \varepsilon}\right)$ for $\varepsilon = \log_2(3/2)$ and it's the first case
  - thus, $T(n) = \Theta\left(n^{\log_2 3}\right)$
- $T(n) = \frac{1}{2}T(n/2) + n^2$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^2$
  - $a = 4,\ b = 2,\ f(n) = n^2$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Theta(n^2 . \log^0 n)$ for $k = 0$ and it's the second case
  - thus, $T(n) = \Theta(n^2 . \log n)$
- $T(n) = 3T(n/2) + n$
  - $a = 3,\ b = 2,\ f(n) = n$, thus $n^{\log_b a} = n^{\log_2 3}$
  - since $f(n) = O\left(n^{\log_2 3 - \varepsilon}\right)$ for $\varepsilon = \log_2(3/2)$ and it's the first case
  - thus, $T(n) = \Theta\left(n^{\log_2 3}\right)$
- $T(n) = \frac{1}{2}T(n/2) + n^2$
  - since $a = \frac{1}{2}$ is not ≥ 1, Master Theorem cannot be applied

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^3$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^3$
  - $a = 4,\ b = 2,\ f(n) = n^3$, thus $n^{\log_b a} = n^2$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^3$
  - $a = 4,\ b = 2,\ f(n) = n^3$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and $4(cn/2)^3 \leq cn^3$ for $\mathrm{c} = 1/2$, it's the third case

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^3$
  - $a = 4,\ b = 2,\ f(n) = n^3$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and $4(cn/2)^3 \leq cn^3$ for $\mathrm{c} = 1/2$, it's the third case
  - thus, $T(n) = \Theta(n^3)$

Worst-Case Analysis

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  - $a = 4,\ b = 2,\ f(n) = n^3$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and $4(cn/2)^3 \leq cn^3$ for $\mathrm{c} = 1/2$, it's the third case
  - thus, $T(n) = \Theta(n^3)$
- $T(n) = 2T(n/2) + n$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^3$
  - $a = 4,\ b = 2,\ f(n) = n^3$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and $4(cn/2)^3 \leq cn^3$ for $\mathrm{c} = 1/2$, it's the third case
  - thus, $T(n) = \Theta(n^3)$
- $T(n) = 2T(n/2) + n$
  - $a = 2,\ b = 2,\ f(n) = n$, thus $n^{\log_b a} = n$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^3$
  - $a = 4,\ b = 2,\ f(n) = n^3$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and $4(cn/2)^3 \le cn^3$ for $\mathrm{c} = 1/2$, it's the third case
  - thus, $T(n) = \Theta(n^3)$
- $T(n) = 2T(n/2) + n$
  - $a = 2,\ b = 2,\ f(n) = n$, thus $n^{\log_b a} = n$
  - since $f(n) = \Theta(n.\log^0 n)$ for $k = 0$ and it's the second case

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^3$
  - $a = 4,\ b = 2,\ f(n) = n^3$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and $4(cn/2)^3 \leq cn^3$ for $\mathrm{c} = 1/2$, it's the third case
  - thus, $T(n) = \Theta(n^3)$
- $T(n) = 2T(n/2) + n$
  - $a = 2,\ b = 2,\ f(n) = n$, thus $n^{\log_b a} = n$
  - since $f(n) = \Theta(n.\log^0 n)$ for $k = 0$ and it's the second case
  - thus, $T(n) = \Theta(n.\log n)$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^3$
  - $a = 4,\ b = 2,\ f(n) = n^3$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and $4(cn/2)^3 \leq cn^3$ for $\mathrm{c} = 1/2$, it's the third case
  - thus, $T(n) = \Theta(n^3)$
- $T(n) = 2T(n/2) + n$
  - $a = 2,\ b = 2,\ f(n) = n$, thus $n^{\log_b a} = n$
  - since $f(n) = \Theta(n.\log^0 n)$ for $k = 0$ and it's the second case
  - thus, $T(n) = \Theta(n.\log n)$
- $T(n) = 2T(4n/3) + n^2$

Worst-Case Analysis

Master Theorem

- $T(n) = 4T(n/2) + n^3$
  - $a = 4,\ b = 2,\ f(n) = n^3$, thus $n^{\log_b a} = n^2$
  - since $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ and $4(cn/2)^3 \leq cn^3$ for $\mathrm{c} = 1/2$, it's the third case
  - thus, $T(n) = \Theta(n^3)$
- $T(n) = 2T(n/2) + n$
  - $a = 2,\ b = 2,\ f(n) = n$, thus $n^{\log_b a} = n$
  - since $f(n) = \Theta(n.\log^0 n)$ for $k = 0$ and it's the second case
  - thus, $T(n) = \Theta(n.\log n)$
- $T(n) = 2T(4\,n/3) + n^2$
  - since $\mathrm{b} = \frac{3}{4}$ is not > 1, Master Theorem cannot be applied