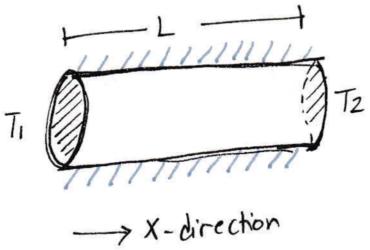
CHAPTER - 2 INTRODUCTION TO CONDUCTION

* Recall that conduction is the transport of energy in a medium of a temperature gradient and the physical medium has molecular activity.

* What form does the Fourier's Law take in different geometries?
* HOW does k depend on the nature of the medium (k = constant, general assumption)
* How do we develop the heat equation? [Heat equation means energy balance]

Suppose that the cylindrical rod below is open to heat transfer at its two ends. But the lateral surface is insulated.



$T_1 > T_2$ Only end surfaces transfer heat.

assumptions: steady-state conduction

Fourier's Law: $q''_x = -k \dfrac{dT}{dx} = -k \dfrac{T_2 - T_1}{L}$

$$q''_x = k \frac{T_1 - T_2}{L} = \frac{k \,\Delta T}{L}$$

$T_1 - T_2 = \Delta T$

$q''_x = k \dfrac{\Delta T}{L}$ Heat transfer rate per unit area by conduction in x-direction (Heat flux) $\Rightarrow q''_x = k \dfrac{\Delta T}{L}$

q''_x varies directly with k and ΔT

q''_x varies inversely with L

k: thermal conductivity will be given to you in the questions. k $\left(\dfrac{W}{m \cdot K}\right)$

k is assumed constant generally.

k is a transport property and provides an indication of the rate at which energy is transferred by the diffusion process.

- k may change according to the coordinates ($k_x \neq k_y = k_z$)

But for an isotropic material → $k_x = k_y = k_z$

- $k_{solid} > k_{liquid} > k_{gas}$ [this difference is mainly due to the intermolecular spacing for the states of matter]

- k may change with respect to temperature.

* In our analysis of ~~heat~~ transfer problems, it will be necessary to use several properties of matter.

Transport properties

k: conductivity $\left(\frac{W}{m \cdot K}\right)$

ν : kinematic viscosity $\left(\frac{m^2}{s}\right)$

$$\nu = \frac{\mu}{\rho} = \frac{\frac{kg}{m \cdot s}}{\frac{kg}{m^3}} = \frac{m^2}{s}$$

μ: dynamic viscosity

Thermodynamic Properties

These properties are related to the equilibrium state of a system.

ρ : density $\left(\frac{kg}{m^3}\right)$

C_p: heat capacity at constant P $\left(\frac{kJ}{kg \cdot K}\right)$

$$\rho \cdot C_p = \frac{kg}{m^3} \cdot \frac{kJ}{kg \cdot K} = \frac{kJ}{m^3 \cdot K}$$

$\rho \cdot C_p$ is called the volumetric heat capacity.

Volumetric heat capacity is the material's ability to store thermal energy.

$\rho \cdot C_p > 1 \frac{MJ}{m^3 \cdot K}$ → very good thermal storage media

$\rho \cdot C_p \approx 1 \frac{MJ}{m^3 \cdot K}$

* In heat transfer problems, the ratio of the thermal conductivity to the volumetric heat capacity is an important property called α : thermal diffusivity $\left(\frac{m^2}{s}\right)$

$$\alpha = \frac{k}{\rho \cdot C_p} = \frac{(W/\cancel{m} \cdot \cancel{K})}{\left(\frac{\cancel{kg}}{\cancel{m^3}_{m^2}} \cdot \frac{\cancel{kJ}}{\cancel{kg} \cdot \cancel{K}}\right)} = \frac{m^2}{s}$$

α measures the material to conduct thermal energy relative to its ability to store thermal energy.

* Materials of large α will respond more quickly to changes in their thermal environment, whereas materials of small α will respond more sluggishly, taking longer to reach a new equilibrium condition.

Heat Diffusion Equation (Heat Equation) (Conduction)

* We wish to know the temperature distribution within a material.
* If you know two different temperatures of two different points within a material, you can use Fourier's law to find the heat flux.
* We mean if temperature distribution within a material is known, then heat transfer flux can be determined by using Fourier's Law.

Temperature distribution ⇒ T is a function of X

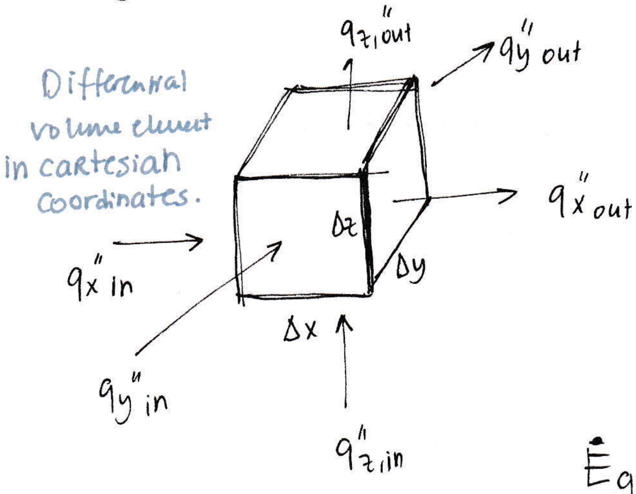
ex: $T = ax^2 + bx + c$

So having temperature distribution is advantageous.

If you don't have temperature distribution, then you derive it by using heat equation (heat conduction equation).

Heat equation which is simply the conservation of energy for a system in which only conduction occurs should be derived accordingly:

1. Determine the geometry → which coordinate system do you choose?
2. Define a differential control volume (differential volume element) as the system
3. Establish energy balance for that differential volume element.



Energy balance general form:

$$\dot{E}_{in} - \dot{E}_{out} + \underbrace{\dot{E}_{gen}}_{\text{generation term}} = \underbrace{\dot{E}_{st}}_{\text{storage (accumulation) term}} \quad (\text{UNIT}: W)$$

$\dot{E}_{st} = 0$ [under S.S conditions]

$\dot{E}_{gen}$: thermal energy generated per unit time ⇒ $\dot{E}_{gen} = \underbrace{\dot{q}_{gen}}_{\text{energy generation per unit volume}} [\Delta x \cdot \Delta y \cdot \Delta z]$

$\dot{E}_{st}$: thermal energy stored per unit time ⇒ $\dot{E}_{st} = \rho\, c_p \dfrac{\partial T}{\partial t} [\Delta x \cdot \Delta y \cdot \Delta z]$

Please understand how heat equation (below) is derived step-by-step

⇓

$$\left[q''_{x\,in} - q''_{x\,out}\right][\Delta y \Delta z] + \left[q''_{y\,in} - q''_{y\,out}\right][\Delta x \cdot \Delta y] + \left[q''_{z\,in} - q''_{z\,out}\right][\Delta x \cdot \Delta y]$$

$$+ \dot{q}_{gen}[\Delta x \cdot \Delta y \cdot \Delta z] = \rho\, c_p \frac{\partial T}{\partial t}[\Delta x \cdot \Delta y \cdot \Delta z]$$

Divide both sides by $\Delta x \cdot \Delta y \cdot \Delta z$ and take limit $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$

$$-\frac{\partial q''_x}{\partial x} - \frac{\partial q''_y}{\partial y} - \frac{\partial q''_z}{\partial z} + \dot{q}_{gen} = \rho\, c_p \frac{\partial T}{\partial t}$$

$$q''_x = -k\frac{\partial T}{\partial x} \qquad q''_y = -k\frac{\partial T}{\partial y} \qquad q''_z = -k\frac{\partial T}{\partial z} \qquad [k_x = k_y = k_z = k]$$

$$\boxed{+\frac{\partial}{\partial x}\left(+k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(+k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(+k\frac{\partial T}{\partial z}\right) + \dot{q}_{gen} = \rho\, c_p \frac{\partial T}{\partial t}}$$

(2.19) in the book.

Heat diffusion (conduction) equation is cartesian coordinate system.

Since k is considered constant

The equation can be organized as follows:

$$k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \dot{q}_{gen}\right] = \rho C_p \frac{\partial T}{\partial t}$$

Dividing both sides by k

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}_{gen}}{k} = \underbrace{\frac{\rho C_p}{k}}_{\frac{1}{\alpha}} \frac{\partial T}{\partial t} \qquad \left[\text{Remember } \alpha = \frac{k}{\rho \cdot C_p}\right]$$

$$\underbrace{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}}_{\text{conduction terms in cartesian coordinate}} + \underbrace{\frac{\dot{q}_{gen}}{k}}_{\text{generation term}} = \underbrace{\frac{1}{\alpha}\frac{\partial T}{\partial t}}_{\text{storage/ (accumulation) term}}$$

(2.21 in the text book)

This last form is the most simplified version of heat equation in Cartesian coordinates. As mentioned, this equation is used to determine temperature distribution within a system in which heat transfer occurs only by conduction.

Assumption (1): if you assume $T = f(x)$ only $\Rightarrow \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial z^2} = 0$

Assumption (2): if there is no heat generation $\Rightarrow \dot{q}_{gen} = 0$

Assumption (3): The system is at steady-state $\Rightarrow \frac{\partial T}{\partial t} = 0$

temperature change with respect to time is equal to zero.

the equation takes this form:

$$\frac{\partial^2 T}{\partial x^2} = 0 \rightarrow \frac{\partial T}{\partial x} = m \qquad \boxed{T = mx + n}$$

So you obtained temperature distribution which means you have the equation T as a Function of x $[T = f(x)]$

m and n are arbitrary constants.

Once m and n are determined, temperature distribution will be obtained.

Such as you have $T = 3x^2 - 4x + 5$ then the heat flux at $x=0$ ($\dot{q}_x''$)

T (K)

x (m)

$k = 3\ W/m \cdot K$

From Fourier's Law: $\dot{q}_x'' = -k \left.\frac{dT}{dx}\right|_{x=0} = -k\left[-6x - 4\right]_{x=0}$

$= -3\left[-6 \cdot 0 - 4\right] = 12\ \frac{W}{m^2}$

How are the arbitrary constants ; m and n found ?

We need boundary conditions or an initial condition to get the temperature distribution completely [Refer to Table 2.2 on page 83 of the text book] :

Boundary conditions for the heat diffusion equation at the surface $(x=0)$

1- Constant surface temperature

$$T(x=0) = T_s \text{ [surface temperature]}$$

2 - Constant surface heat flux

a) Finite heat flux

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q_s''$$

b) Adiabatic or insulated surface

$$\frac{\partial T}{\partial x} = 0 \quad [q_s''=0]$$

3 - Convection surface condition

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h\left[T_\infty - T(x=0)\right]$$

In the temperature distribution $T = mx+n$, we have 2 arbitrary constants. It means we need 2 boundary conditions to determine m and n .

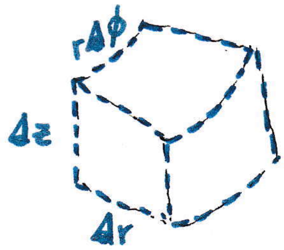
What about the heat equation in the cylindrical and spherical coordinates ?

Eqn. 2.26 of the text book

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}_{gen} = \rho C_p \frac{\partial T}{\partial t}$$

Coordinates are : r, ϕ and z

$r\Delta\phi$, Δz and Δr are differential lengths.



- Differential volume element in cylindrical coordinates

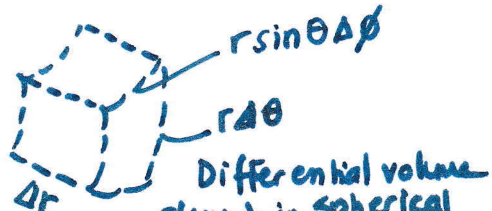
$$q_r'' = -k\frac{\partial T}{\partial r} \qquad q_\phi'' = -\frac{k}{r}\frac{\partial T}{\partial \phi} \qquad q_z'' = -k\frac{\partial T}{\partial z} \quad \text{(Heat flux equations)}$$

Eqn. 2.29 of the text book

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{q}_{gen} = \rho C_p \frac{\partial T}{\partial t}$$

Coordinates are : r, ϕ and θ

$r\sin\theta\Delta\phi$, $r\Delta\theta$ and Δr are differential lengths.



Differential volume element in spherical coordinates

$$q_r'' = -k\frac{\partial T}{\partial r} \qquad q_\theta'' = -\frac{k}{r}\frac{\partial T}{\partial \theta} \qquad q_\phi'' = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi} \quad \text{(Heat flux equations)}$$