CHAPTER 3 – ONE DIMENSIONAL STEADY-STATE CONDUCTION

In this chapter, we treat situations for which heat is transferred by diffusion (conduction) under one dimensional, steady-state conditions.

Case-1 : 1-D, Steady-state conduction with no heat generation :

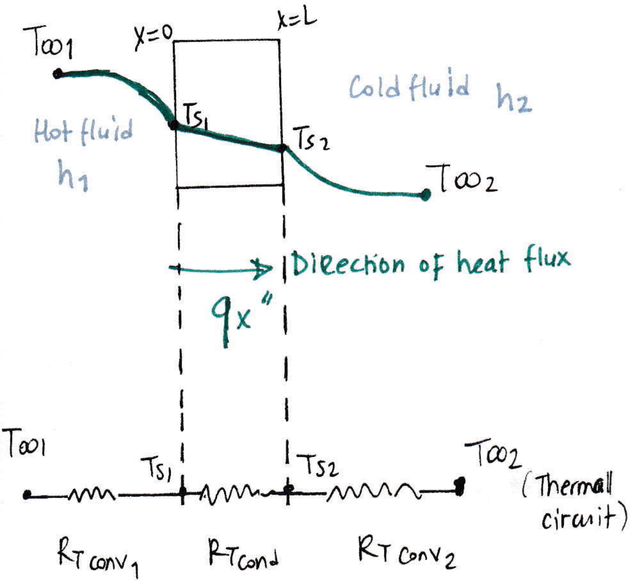
Planar wall example [this is the most typical example in cartesian coordinate system]

Heat equation simplifies to : $\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$ under the conditions: S.S. no heat generation 1-D conduction

Heat equation tells us that $\frac{dT}{dx} = C_1 = \text{Constant}$

$T = C_1 x + C_2$ (T varies linearly with x)

→ a Plane wall that separates two fluids



On the left, we have a plane wall which separates a hot fluid and a cold fluid.
Hot fluid is at $T_{\infty 1}$ and has a convection coefficient of h_1.
Cold fluid is at $T_{\infty 2}$ and has a convection coefficient of h_2.

Hot fluid, cold fluid and the planar wall; each media possess thermal resistance to heat transfer.

Hot fluid shows $R_{T,conv_1}$

Cold fluid shows $R_{T,conv_2}$

Planar wall shows $R_{T,cond}$

[R denotes resistance, T denotes thermal]

R_T : thermal resistance

Recall the heat equation $\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$ $\quad \frac{dT}{dx} = C_1 = \text{constant}$, then $\boxed{q_x'' = -k\frac{dT}{dx} = \text{constant}}$

meaning that heat flux, q_x'' is independent of x.

k = Constant [k is not a function of temperature or x], so you can take it out of paranthesis.

$$\frac{d}{dx}\left(\frac{dT}{dx}\right)=0 \rightarrow \frac{dT}{dx}=C_1 \longrightarrow T = C_1 x + C_2$$

To obtain integration constants; C_1 and C_2 we need two Boundary Conditions (BCs) which are surface BCs :

at $x=0$ $\quad T=T_{s1} \longrightarrow T_{s1} = C_1 \cdot 0 + C_2 \longrightarrow \boxed{C_2 = T_{s1}}$

$x=L$ $\quad T=T_{s2} \longrightarrow T_{s2} = C_1 \cdot L + C_2 \longrightarrow T_{s2} = C_1 \cdot L + T_{s1}$

$$\boxed{C_1 = \frac{T_{s2}-T_{s1}}{L}}$$

So T becomes equal to $\boxed{T = \frac{T_{s2}-T_{s1}}{L}x + T_{s1}}$ → this is the temperature distribution within plane wall.

$$q''_x = -k\,\frac{dT}{dx} = -k \cdot \frac{T_{s2} - T_{s1}}{L} = \frac{k\,(T_{s1} - T_{s2})}{L} \quad \text{[Heat flux]}$$

$$\frac{dT}{dx} \sim \frac{T_{s2} - T_{s1}}{L}$$

Heat transfer rate by conduction in plane wall.

$$q_x = q''_x\,[\text{Area normal to the heat transfer}] = k\,A\,\frac{T_{s1} - T_{s2}}{L}$$

Thermal Resistance

Heat transfer Rate $\Rightarrow q_x = k\,A\,\dfrac{T_{s1} - T_{s2}}{L}$

reorganize the equation

$$q_x = \frac{T_{s1} - T_{s2}}{\left(\frac{L}{kA}\right)}$$

$T_{s1} - T_{s2}$: driving potential $\Rightarrow$ temperature gradient (ΔT) which leads to heat transfer in the plane wall.

$\boxed{R_{T_{cond}} = \dfrac{L}{kA}}$ $\rightarrow$ thermal resistance to conduction

Heat transfer rate $= \dfrac{\text{Driving Potential}}{\text{Resistance (thermal resistance)}}$ $\rightarrow$ $\boxed{\text{Thermal Resistance} = \dfrac{\text{Driving Potential}}{\text{Heat transfer rate}}}$

similar

Remember electrical conduction $\Rightarrow$ $\boxed{R_e = \dfrac{E_{s1} - E_{s2}}{I}}$ (Ohm's Law)

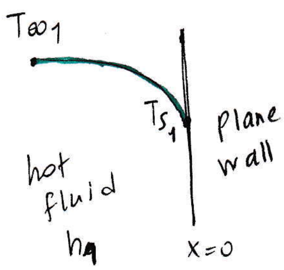
* Thermal circuit representations provide a useful tool for both conceptualizing and qualifying heat transfer problems.

q''_x = Constant [when the system is at s.s, $\dot{q}_{gen} = 0$, conduction occurs in only x-direction]

A = constant [area normal to the heat transfer]

q_x = constant

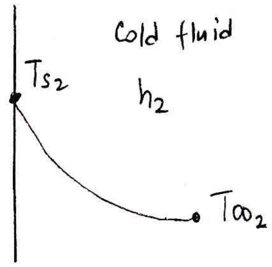
Remember Newton's Law of Cooling $q_x = h\,A\,(T_\infty - T_s)$



$$q_x = h_1\,A\,(T_{\infty 1} - T_{s1}) = \text{constant}$$

$$q_x = \frac{T_{\infty 1} - T_{s1}}{\frac{1}{h_1 A}} = \frac{\text{Driving potential}}{\text{Thermal resistance to convection by hot fluid}} \rightarrow \boxed{R_{T_{conv,1}} = \frac{1}{h_1 A}}$$

Plane wall



$$q_x = h_2 A (T_{s2} - T_{\infty 2})$$

$$q_x = \frac{T_{s2} - T_{\infty 2}}{\frac{1}{h_2 A}} = \frac{\text{Driving potential}}{\text{Thermal resistance to convection by cold fluid}}$$

$$q_x = \frac{T_{\infty 1} - T_{s1}}{\frac{1}{h_1 A}} = \frac{T_{s1} - T_{s2}}{\frac{L}{kA}} = \frac{T_{s2} - T_{\infty 2}}{\frac{1}{h_2 A}}$$

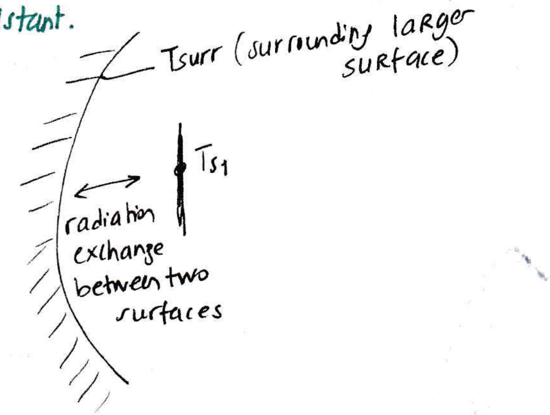
- Since q_x is constant, we can relate thermal resistances by the equation above.
- In terms of overall temperature difference $T_{\infty 1} - T_{\infty 2}$ and the total thermal resistance, R_{tot}, the heat transfer rate, q_x may also be expressed as:

$$\boxed{q_x = \frac{T_{\infty 1} - T_{\infty 2}}{R_{tot}}} \qquad \boxed{\underset{(\Sigma R_t)}{R_{tot}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}}$$

(Eqn. 3-12 of the textbook)

These two equations are very important since if you have the fluid temperatures but do not have surface temperatures (T_{s1} and T_{s2}), you can calculate heat transfer rate by using the total thermal resistant.

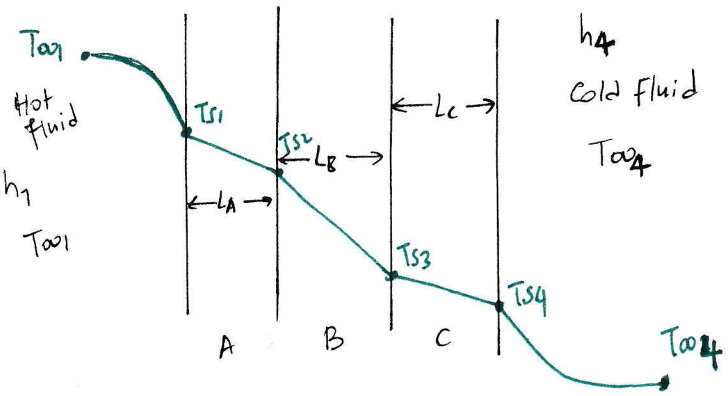
Extra info: $\underbrace{R_{T,\,radiation}}_{\text{thermal resistance to radiation}} = \frac{T_{s1} - T_{surr}}{q_{rad}}$



Composite wall

Equivalent thermal circuits may also be used for more complex systems, such as composite walls. Such walls may involve series or parallel thermal resistances due to layers of different materials.

Suppose that you have composite walls, A, B, and C in series.
$k_A \neq k_B \neq k_C$ (they are different materials)



Under S.S. conditions and no heat generation

$$q_x = \frac{T_{\infty 1} - T_{\infty 4}}{\frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_4 A}}$$

The denominator is ΣR_t total resistance represented by the fluids and each wall in total.

$$q_x = \frac{T_{\infty 1} - T_{s1}}{\frac{1}{h_1 A}} = \frac{T_{s1} - T_{s2}}{\frac{L_A}{k_A \cdot A}} = \frac{T_{s2} - T_{s3}}{\frac{L_B}{k_B \cdot A}} = \frac{T_{s3} - T_{s4}}{\frac{L_C}{k_C A}} = \frac{T_{s4} - T_{\infty 4}}{\frac{1}{h_4 A}}$$

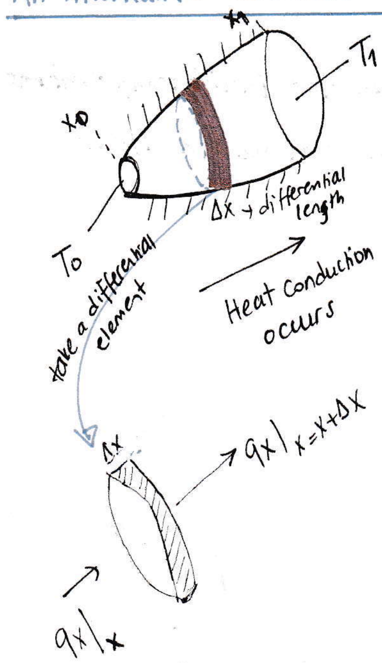
With composite systems, it is much easier to work with an overall heat transfer coefficient U which is defined by an expression analogous to Newton's of cooling.

$q_x = UA\Delta T$ U: overall heat transfer coefficient $q_x = \frac{\Delta T}{\frac{1}{UA}}$

$$U = \frac{1}{R_{tot} \cdot A} = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_4}}$$

$$\boxed{\Sigma R_T = R_{tot} = \frac{1}{UA} = \frac{\Delta T}{q_x}}$$

An Alternative Conduction Analysis



Consider the system on the left.

Lateral surface is insulated (lateral surface is adiabatic)

Ends are open to heat transfer.

Each end is at a different temperature.

No heat generation ($\dot{q}_{gen} = 0$)

The system is at steady-state

However A is changing $\Rightarrow A = f(x)$ or $A(x)$

We know that q_x is constant

$q_x = -k\, A(x) \frac{dT}{dx}$ [Written according to Fourier's Law]

Let's integrate the above equation

$$\boxed{q_x \int_{x_0}^{x} \frac{dx}{A(x)} = -k \int_{T_0}^{T} dT} \rightarrow \text{Solve this when } A = f(x)$$

$$\boxed{q_x \frac{\Delta x}{A} = -k\, \Delta T}$$

where $\Delta x = x_1 - x_0$, $\Delta T = T_1 - T_0$ when A is constant

Radial systems → Cylindrical and spherical coordinate systems

One dimensional heat conduction at steady-state without heat generation will give the following form of heat equation in the cylindrical coordinates:

$\underbrace{\text{One dimensional}}^{\text{only in r-direction}}$

$$\frac{1}{r}\frac{d}{dr}\left(\underset{\substack{\downarrow \\ \text{constant}}}{k} r \frac{dT}{dr}\right) = 0 \rightarrow kr\frac{dT}{dr} = \text{constant} \rightarrow r\frac{dT}{dr} = C_1 \rightarrow \frac{dT}{dr} = \frac{C_1}{r}$$

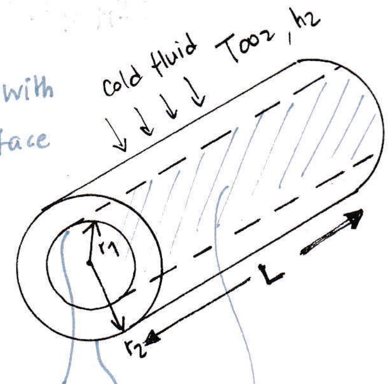
Solution → $\boxed{T = C_1 \ln r + C_2}$

Fourier's Law in r-direction will be $q_r'' = -k\frac{dT}{dr}$

Heat transfer rate across any cylindrical surface in the solid $\quad q_r = -kA\frac{dT}{dr} = -k(2\pi r L)\frac{dT}{dr}$

$A = 2\pi r L$ is the area normal to the direction of heat transfer.

Hollow cylinder with convective surface conditions

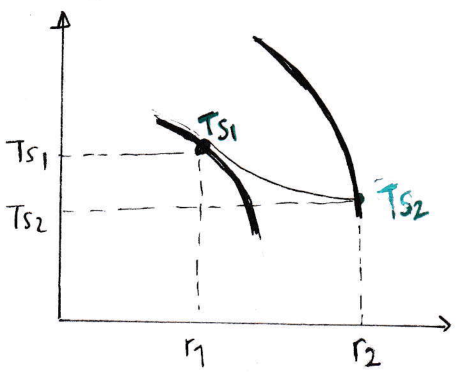


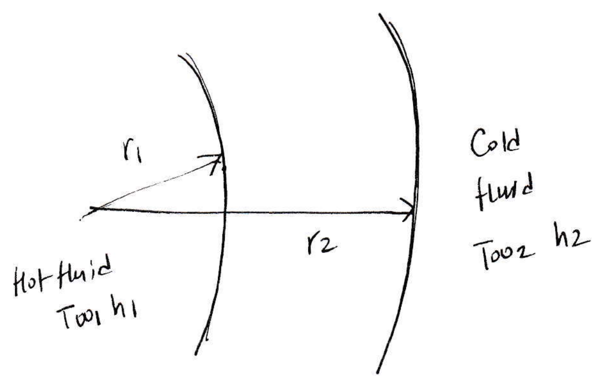
Since $kr\frac{dT}{dr}$ is a constant

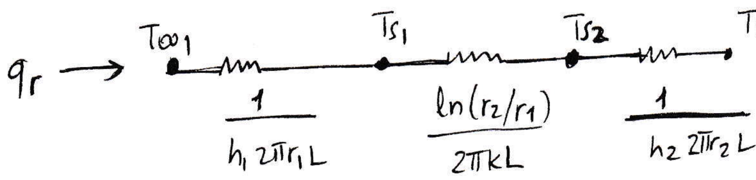
q_r is also constant, meaning that it is not dependent on r. But q_r'' [heat flux] is not constant, it is dependent on r.

In cylindrical coordinate → $q_r \Rightarrow$ constant
$q_r'' \Rightarrow$ not constant
$q_r'' = f(r)$!

Take a closed look on the cylinder and temperature distribution (profile):







Thermal circuit

Resistance terms for convection and conduction in a cylinder

Let's return to the solution of heat equation which gives T distribution in r-direction within a solid cylinder:

$T = C_1 \ln r + C_2$, substitute the typical surface temperature BC's → at $r = r_1$ $T = T_{s1}$
at $r = r_2$ $T = T_{s2}$

$$\left.\begin{aligned} T_1 &= C_1 \ln r_1 + C_2 \\ T_2 &= C_1 \ln r_2 + C_2 \end{aligned}\right] \text{ Solving simultaneously: } \boxed{T(r) = \frac{T_{s1} - T_{s2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s2}}$$

* q_r (Heat transfer rate) $= \dfrac{2\pi L k \,(T_{s1} - T_{s2})}{\ln(r_2/r_1)} = \dfrac{T_{s1} - T_{s2}}{\dfrac{\ln(r_2/r_1)}{2\pi L k}}$ → R_T, cond in a cylinder

* $R_{T,\,cond} = \dfrac{\ln(r_2/r_1)}{2\pi L k}$ → For a cylinder $R_{T,\,cond}$ also depends on the cylinder length (L)

$$q_r = \frac{T_{\infty 1} - T_{\infty 2}}{\dfrac{1}{2\pi r_1 L h_1} + \dfrac{\ln(r_2/r_1)}{2\pi L k} + \dfrac{1}{2\pi r_2 L h_2}}$$

* R_T, conv for hot fluid side ⇒ $R_{T,\,conv} = \dfrac{T_{\infty 1} - T_{s1}}{q_r}$ → $R_{T,\,conv} = \dfrac{1}{2\pi r_1 L h_1}$

Review the equations for a sphere <u>on your own</u>.

Review <u>Table 3.3</u> on page 132 of the text-book.
