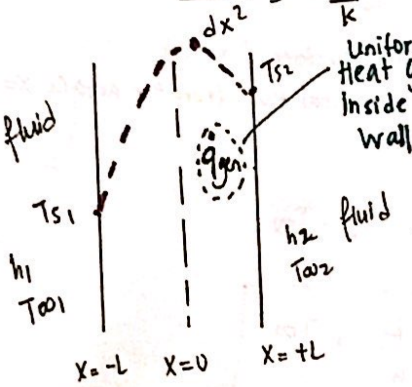
Case 2 : One-dimensional steady-state Conduction with heat generation

Consider the plane wall in which there is uniform heat generation per unit volume $\dot{q}\left(\frac{W}{m^3}\right)$, the surfaces are maintained at T_{S1} and T_{S2}. For constant thermal conductivity, k, the appropriate form of the equation is:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}_g}{k} = 0 \quad \left[\begin{array}{l} \text{1-D Conduction only in x-direction} \\ \text{Steady-state conduction} \\ \text{Uniform heat generation} \rightarrow \dot{q}_{gen} \end{array}\right]$$

$$\frac{d^2T}{dx^2} = -\frac{\dot{q}_g}{k} \rightarrow \frac{dT}{dx} = -\frac{\dot{q}_g x}{k} + C_1 \rightarrow \boxed{T(x) = \frac{-\dot{q}_g}{2k}x^2 + C_1x + C_2}$$

Temperature distribution



* Boundary conditions → $x=-L \quad T=T_{S1}$
(Asymmetrical BCs) $\quad x=+L \quad T=T_{S2}$

Substituting BCs will give: $T_{S1} = -\frac{\dot{q}_g}{2k}L^2 - C_1L + C_2$

$$T_{S2} = -\frac{\dot{q}_g}{2k}L^2 + C_1L + C_2$$

Solving the equations simultaneously will give

$$\boxed{T(x) = \frac{\dot{q}_g L^2}{2k}\left[1 - \frac{x^2}{L^2}\right] + \frac{T_{S2}-T_{S1}}{2}\frac{x}{L} + \frac{T_{S1}+T_{S2}}{2}}$$

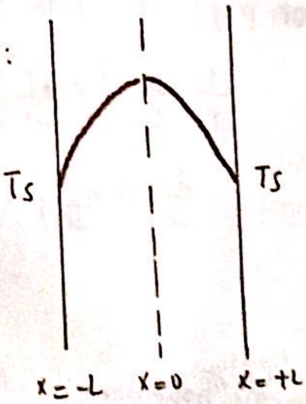
$$\text{Heat flux} = q''_x = -k\frac{dT}{dx} = -k\left[\frac{\dot{q}_g L^2}{\cancel{2k}}\left(-\frac{\cancel{2}x}{L^2}\right) + \frac{T_{S2}-T_{S1}}{2L}\right] = \dot{q}_g L^2 x - \left(\frac{T_{S2}-T_{S1}}{2L}\right)k$$

so as you see heat flux is not independent on x anymore under conditions with heat generation.

* If you have symmetrical BCs, according to the scheme on the right:

$x=0 \quad T=T_o$

$x=\pm L \quad T=T_s$



Temperature distribution will take the following form:

$$\boxed{T(x) = \frac{\dot{q}_g L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s} \quad (1)$$

If you have symmetrical BCs, the maximum temperature T_o exists at the midplane $\Rightarrow$ $x=0 \quad T=T_o$

$$T_o = \frac{\dot{q}_g L^2}{2k} + T_s \rightarrow \boxed{T_o - T_s = \frac{\dot{q}L^2}{2k}} \quad (2)$$

Taking the ratio of eqns (1) and (2) will give

$$\boxed{\frac{T(x)-T_s}{T_0-T_s} = 1-\frac{x^2}{L^2}}$$

Temperature distribution for a system:
- at S.S
- cartesian coordinates
- 1-D conduction [only x- direction]
- uniform heat generation [$\dot{q}_{gen}$]
- Symmetrical BCs

$$T(x) = \left(1-\frac{x^2}{L^2}\right)\left(T_0-T_s\right)$$

$$T(x) = \left(1-\frac{x^2}{L^2}\right)\left(\frac{\dot{q}L^2}{2k}\right)+T_s$$

Let's say you have no idea about T_s, you can use the following BC to relate T_s to T_∞

$$-k\left.\frac{dT}{dx}\right|_{x=L} = h\left[T_s-T_\infty\right] \rightarrow$$

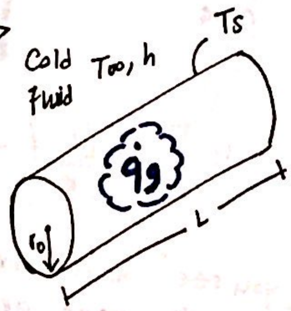
Conductive heat flux at the surface $(x=L)$ is equal to the convective heat flux from the surface $x=L$

$$-\cancel{k}\left[-\frac{\dot{q}_{gen}\cancel{L^2}}{\cancel{2k}}\cdot\frac{\cancel{2}L}{\cancel{L^2}}\right] = h\left[T_s-T_\infty\right] \rightarrow \frac{+\dot{q}_{gen}L}{h} = T_s-T_\infty$$

$$\boxed{T_s = \frac{\dot{q}L}{h}+T_\infty}$$

Radial systems in which 1-D conduction occuring under S.S. Conditions with heat generation

A cylindrical rod with uniform heat generation $\Rightarrow$
The cylindrical rod is at steady-state.
1-D conduction occurs within the system (only in r-direction)



Refer to heat (diffusion / conduction) equation on page 138, Eqn. 3.54 in the text-book:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}_g}{k} = 0$$

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\dot{q}_g r}{k} \rightarrow r\frac{dT}{dr} = -\frac{\dot{q}_g r^2}{2k}+C_1 \rightarrow \frac{dT}{dr} = -\frac{\dot{q}_g r}{2k}+\frac{C_1}{r}$$

$$\boxed{T(r) = -\frac{\dot{q}r^2}{4k}+C_1\ln r+C_2}$$

Apply Symmetry boundary conditions

$r=0 \quad \frac{dT}{dr}=0 \quad T=T_0$

$r=r_0 \quad T=T_s$

$$T(r) = T_s+\frac{\dot{q}r_0^2}{4k}\left[1-\left(\frac{r}{r_0}\right)^2\right]$$

$$T_0 = T_s+\frac{\dot{q}r_0^2}{4k}$$

$\xrightarrow{\text{Combining}}$

$$\boxed{\frac{T(r)-T_s}{T_0-T_s} = 1-\left(\frac{r}{r_0}\right)^2}$$

Temperature distribution within a cylindrical rod with uniform heat generation.