NUCLEAR BINDING ENERGY

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Nuclear Binding Energy

The total mass of a nucleus is less than the sum of the masses of its individual nucleons. Therefore, the rest energy of the bound system (the nucleus) is less than the combined rest energy of the separated nucleons. This difference in energy is called the *binding energy* of the nucleus and can be interpreted as the energy that must be added to a nucleus to break it apart into its components.

The mass energy $m_N c^2$ of a certain nuclide is its atomic mass energy $m_A c^2$ less the total mass energy of Z electrons and the total *electronic* binding energy:

$$m_N c^2 = m_A c^2 - Z m_e c^2 + \sum_{i=1}^{Z} B_i \qquad (1)$$

where B_i is the binding energy of the ith electron.

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Electronic binding energies are of order 10-100 keV in heavy atoms, while atomic mass energies are of order *A* x 1000 MeV; thus we can neglect the last term of Eq. 1.

The *binding energy B* of a nucleus is the difference in mass energy between a nucleus ${}_{Z}^{A}X_{N}$ and its constituent *Z* protons and *N* neutrons:

$$B = \left\{ Zm_p + Nm_n - \left[m\left({}^{A}X \right) - Zm_e \right] \right\} c^2 \qquad (2)$$

Grouping the *Z* proton and electron masses into *Z* neutral hydrogen atoms, we can rewrite Eq. 2

$$B = \left[Zm\left({}^{1}H \right) + Nm_n - m\left({}^{A}X \right) \right] c^2 \qquad (3)$$

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With the masses generally given in atomic mass units, it is convenient to include the unit conversion factor in c^2, thus: $c^2 = 931.50$ MeV/u.

The reported atomic masses of different isotopes of stable elements are on the physical atomic scale of ^{12}C. Masses of the isotopes differ very slightly from integral numbers. This small variation from whole numbers was expressed in terms of a quantity called the packing fraction, f, and is defined as

$$f = \frac{atomic\ mass\ of\ the\ isotope - mass\ number}{mass\ number} = \frac{M\left(A, Z\right) - A}{A} \tag{4}$$

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where *M(A,Z)* is the actual mass of a nuclide on the physical atomic scale of C^{12} (or O^{16}), and A is the mass number $Z+N$, Z and N being the number of protons and neutrons, respectively. The numerator in Eq. 4, $M(A,Z)-A=Af$, is called the mass defect.

Other useful and interesting properties that are often tabulated are the neutron and proton separation energies. The *neutron separation energy* S_n is the amount of energy that is needed to remove a neutron from a nucleus ${}_Z^AX_N$, equal to the difference in binding energies between ${}_Z^AX_N$ and ${}_Z^{A-1}X_{N-1}$:

$$S_n = B\left({}_Z^AX_N\right) - B\left({}_Z^{A-1}X_{N-1}\right) = \left[m\left({}_Z^{A-1}X_{N-1}\right) - m\left({}_Z^AX_N\right) + m_n\right]c^2 \qquad (5)$$

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In a similar way we can define the *proton separation energy* S_p as the energy needed to remove a proton:

$$S_p = B\left({}^{A}_{Z}X_N \right) - B\left({}^{A-1}_{Z-1}X_N \right) = \left[m\left({}^{A-1}_{Z-1}X_N \right) - m\left({}^{A}_{Z}X_N \right) + m\left({}^{1}H \right) \right] c^2 \qquad (6)$$

The hydrogen mass appears in this equation instead of the proton mass, since we are always working with *atomic* masses; you can see immediately how the *Z* electron masses cancel from Equations 6 and 7.

Since the binding energy increases more or less linearly with *A*, it is general practice to show the average binding energy per nucleon, *B/A*, as a function of *A* (See Figure 3.16 in Introductory Nuclear Physics by Kenneth S. Krane to see the variation of *B/A* with nucleon number)

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The average binding energy per nucleon is obtained by dividing the total binding energy of the nucleus by the mass number A.

$$B/A = \left[Zm\left({}^{1}H \right) + Nm_n - m\left({}^{A}X \right) \right] c^2 / A \qquad (7)$$

Several remarkable features are immediately apparent from Figure 3.16 (Introductory Nuclear Physics by Kenneth S. Krane)

- First of all, the curve is relatively constant except for the very light nuclei. The average binding energy of most nuclei is, to within 10 %, about 8 MeV per nucleon.
- Second, we note that the curve reaches a peak near A = 60, where the nuclei are most tightly bound.

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This suggests we can “gain” (that is, release) energy in two ways-below A = 60, by assembling lighter nuclei into heavier nuclei, or above A = 60, by breaking heavier nuclei into lighter nuclei. In either case we “climb the curve of binding energy” and liberate nuclear energy; the first method is known as *nuclear fusion* and the second as *nuclear fission*.

Attempting to understand this curve of binding energy leads us to the *semiempirical mass formula*, in which we try to use a few general parameters to characterize the variation of B with A.

The most obvious term to include in estimating B/A is the constant term, since to lowest order $B \sim A$.

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Specific Nuclear Binding Energy:

The contribution to the binding energy from this "volume" term is thus $B = a_v A$ where a_v , is a constant to be determined, which should be of order 8 MeV. This linear dependence of B on A is in fact somewhat surprising, and gives us our first insight into the properties of the nuclear force.

If every nucleon attracted all of the others, then the binding energy would be proportional to $A(A - 1)$, or roughly to A^2. Since B varies linearly with A, this suggests that each nucleon attracts only its closest neighbors, and *not* all of the other nucleons.

From electron scattering we learned that the nuclear density is roughly constant, and thus each nucleon has about the same number of neighbors; each nucleon thus contributes roughly the same amount to the binding energy.

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An exception to the above argument is a nucleon on the nuclear surface, which is surrounded by fewer neighbors and thus less tightly bound than those in the central region. These nucleons do not contribute to B quite as much as those in the center, and thus $B = a_v A$ overestimates B by giving full weight to the surface nucleons. We must therefore subtract from B a term proportional to the nuclear surface area.

Surface-Tension Effect:

The rapid decrease in the value of the binding energy per nucleon at small A can be explained as the surface-tension effect if the nucleus is viewed as a drop of liquid.

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Surface-Tension Effect:

The nucleons deep inside the nucleus are attracted from every side by the neighboring nucleons while those on the surface are attracted only from one side. This leads to a small value of the binding energy for the surface nucleons. This effect is greater for nuclei with small A because a greater fraction of the nucleons is near the surface as compared to the nuclei with large A. If R is the radius of the nucleus, and S is the coefficient of surface tension, the surface energy E_s, is given by

$$E_s = 4\pi R^2 S = 4\pi\left(r_0 A^{1/3}\right)^2 S = \left(4\pi r_0^2 S\right) A^{2/3} = a_s A^{2/3} \qquad (8)$$

Thus the surface nucleons contribute to the binding energy a term of the form $-a_s A^{2/3}$ (where a_s is approximately equal to 16.80 MeV.

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Coulomb Effect:

The drop in the binding energy curve at large values of A can be explained by the coulomb effect. According to Coulomb's law, the protons inside the nucleus will repel each other, decreasing the binding energy or increasing the mass of the nucleus. The repulsive Coulomb force results in two consequences:

i. The mean binding energy per nucleon will drop as A increase. This is apparent from Figure 3.16 (Introductory Nuclear Physics by Kenneth S. Krane), which shows a gradual drop in B/A at higher values of A.

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Coulomb Effect:

ii. The locus of the stable nuclei should depart from the line N/Z=1 towards the direction of a higher number of neutrons as shown in Fig. 5.7 in Fundamentals of Nuclear Physics by Atam P. Arya. This figure shows the neutron number N versus the proton number Z. Two smooth curves, one through the stable isotopes and the other for N=Z , have been drawn. It is clear from the figure that for low values of N and Z, the stable isotopes have N/Z=1. For the heavier elements, the stability curve gradually departs from the N/Z=1 line, reaching a value of N/Z=1.6 for A=238.

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Coulomb Effect:

The total Coulomb-energy contributed to the binding-energy curve may be calculated in the following manner. We again assume the liquid-drop model of the nucleus even though the drop has a charge of Ze, where Z is the number of protons inside the nucleus, and e is the charge of each proton.

Furthermore, if it is assumed that the charge Ze is uniformly distributed throughout the sphere, the charge density ρ is given by

$$\left(\frac{4\pi}{3}R^3\right)\rho = Ze \qquad or \qquad \rho = \frac{3Ze}{4\pi R^3} \qquad (9)$$

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Coulomb Effect:

The total electrostatic energy, E , of this uniform spherical charge distribution is given by

$$E = \int_0^R \frac{\left(4\pi r^3 \rho / 3\right)\left(\rho 4\pi r^2 dr\right)}{r} \qquad (10)$$

where r is the radial distance from the center of the nucleus and R is the radius of the nucleus. Integrating Eq. 10

$$E = 16\pi^2 \rho^2 R^5 / 15 \qquad (11)$$

And substituting the value of ρ from Eq. 9 into Eq. 11, one gets

$$E = 3Z^2 e^2 / 5R \qquad (12)$$

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Coulomb Effect:

Eq. 12 needs a correction term, because expression for E includes an extra amount of fictitious self-energy of each proton resulting from the assumption that the proton is spread over the whole volume. This self-energy for a proton from Eq. 12 is $3e^2/5R$ and for Z protons is $Z(3e^2/5R)$. Subtracting $Z(3e^2/5R)$ from E, we get the total Coulomb energy E_c, given by

$$E_c = \frac{3}{5}\frac{e^2}{R}Z^2 - \frac{3}{5}\frac{e^2}{R}Z \quad or \quad E_c = \frac{3}{5}\frac{e^2}{R}Z\left(Z-1\right) = \frac{6}{5}\frac{e^2}{R}\frac{Z\left(Z-1\right)}{2} \qquad (13)$$

The number of the proton-proton pairs in a nucleus of atomic number Z (because each of the Z protons interacts with the other (Z-1) protons) is Z(Z-1)/2.

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Coulomb Effect:

The factor of ½ comes in because each pair is counted twice. If $Z>>1$, then $Z(Z-1) \approx Z^2$ and Eq. 13 reduces to

$$E_c = \frac{3e^2 Z^2}{5R} \qquad (14)$$

Substitute for $R=r_0A^{1/3}$ in Eq. 14

$$E_c = \frac{3e^2}{5r_0}\frac{Z^2}{A^{1/3}} = a_c \frac{Z^2}{A^{1/3}} \qquad (15)$$

Where a_c is a constant and equal to 0.72 MeV for $r_0=1.2$ fm.

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Odd-Even Effect:

In addition to other factors, the total binding-energy of a nucleus is determined not only by the ratio of the number of protons and neutrons, but also by weather these numbers are odd or even.

The most stable nuclei tend to have an even number of both protons and neutrons. The least stable nuclei are the odd-odd type. The stabilities of the even-odd and odd-even types of nuclei are almost identical and lie intermediate between the other two.

When we have an odd number of nucleons (odd Z and even N, or even Z and odd N) , this term does not contribute.

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Odd-Even Effect:

However, when both Z and *N* are odd, we gain binding energy by converting one of the odd protons into a neutron (or vice versa) so that it can now form a pair with its formerly odd partner.

There are only four nuclei with odd *N* and *Z* (^{2}H, ^{6}Li, ^{10}B, ^{14}N), but 167 with even *N* and *Z*.

This pairing energy δ is usually expressed as $+\ a_p A^{-3/4}$ for *Z* and *N* even, $-\ a_p A^{-3/4}$ for *Z* and *N* odd, and zero for A odd.

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Pairing of Nucleons (Symmetry Term):

We also note that stable nuclei have $Z = A / 2$. If our binding energy formula is to be realistic in describing the stable nuclei that are actually observed, it must take this effect into account. This term is very important for light nuclei, for which Z = A/2 is more strictly observed. For heavy nuclei, this term becomes less important, because the rapid increase in the Coulomb repulsion term requires additional neutrons for nuclear stability. A possible form for this term, called the symmetry term because it tends to make the nucleus symmetric in protons and neutrons, is $-a_{sym}(A - 2Z)^2/A$ which has the correct form of favoring nuclei with $Z = A/2$ and reducing in importance for large A.

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Combining these five terms we get the complete binding energy:

$$B = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_{sym} \frac{(A-2Z)^2}{A} + \delta \qquad (16)$$

The constants must be adjusted to give the best agreement with the experimental curve of Figure 3.16 (Introductory Nuclear Physics by Kenneth S. Krane). A particular choice of a_v = 15.5 MeV, a_s = 16.8 MeV, a_c = 0.72 MeV, a_{sym} = 23 MeV, a_p = 34 MeV, gives the result shown in Figure 3.17 (Introductory Nuclear Physics by Kenneth S. Krane), which reproduces the observed behavior of *B* rather well.

The Semiempirical Atomic-Mass Formula

By analogy with the liquid drop it is possible to write the semiempirical mass-formula for any atom having mass M(A,Z). The procedure to evolve the mass formula for M(A,Z) is to first write the masses of the constituents of the atom, and then apply the necessary corrections. This results in the familiar Weizsacker semiempirical mass formula. The first term is the mass of the constituents of the atoms, the protons, neutrons, and electrons.

$$M_0 = m_p Z + m_n \left(A - Z \right) + m_e Z = m_H Z + m_n \left(A - Z \right) \qquad (17)$$

Where we have neglected the binding energy of the electron and the proton to form the hydrogen atom.

The Semiempirical Atomic-Mass Formula

Adding mass of the constituents of the atom and all the five terms, we get the following expression for the mass of an atom

$$M(A,Z) = m_H Z + m_n (A-Z) - a_v A + a_s A^{2/3} + a_c \frac{Z(Z-1)}{A^{1/3}} + a_{sym} \frac{(A-2Z)^2}{A} + \delta(A,Z) \qquad (18)$$

where the value of δ (A,Z) is given by - $a_p A^{-3/4}$ for *Z* and *N* even, + $a_p A^{-3/4}$ for *Z* and *N* odd, and zero for A odd.

REFERENCES

1. Introductory Nuclear Physics. Kenneth S. Krane

2. Fundamentals of Nuclear Physics. Atam. P. Arya