Continuity Equation

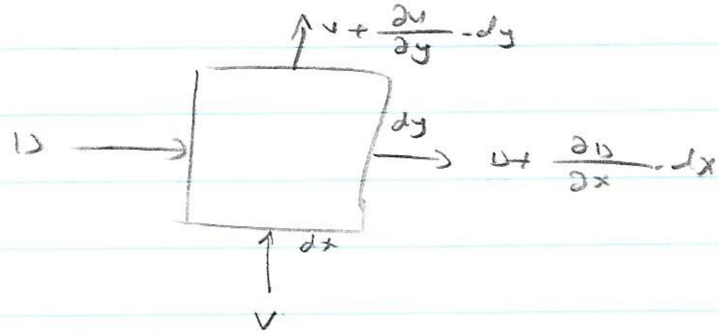
In steady flow, amount of mass within the control volume remains constant.

$$\begin{Bmatrix} \text{rate of mass flow} \\ \text{into the control volume} \end{Bmatrix} = \begin{Bmatrix} \text{rate of mass flow} \\ \text{out of the control volume} \end{Bmatrix}$$

mass flow rate: $\rho \cdot u \cdot A$

u ↓ average velocity

A ↳ cross-sectional area normal to the flow



$$\frac{1}{dx \cdot dy \cdot 1} \left[\rho \cdot u \,(dy \cdot 1) + \rho v \cdot (dx \cdot 1) = \rho \left(u + \frac{\partial u}{\partial x} \cdot dx \right)(dy \cdot 1) + \rho \left(v + \frac{\partial v}{\partial y} \cdot dy \right)(dx \cdot 1) \right]$$

$$\cancel{\frac{\rho u}{dx}} + \cancel{\frac{\rho v}{dy}} = \cancel{\frac{\rho u}{\partial x}} + \frac{\partial u}{\partial x} + \cancel{\frac{\rho v}{dy}} + \frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \longleftarrow \text{conservation of mass}$$

In the case of an axially symmetric flow,

$$\frac{\partial u}{\partial x} + \frac{1}{r} \cdot \frac{\partial}{\partial r}(rv) = 0$$

Momentum Equations

Newton's 2nd law: expression of momentum balance
net force acting on the control volume is equal to the mass x acceleration of the fluid within the control volume.

$$\delta m \cdot a_x = F_{surface,x} + F_{body,x}$$

$$\delta m = \rho\,(dx \cdot dy \cdot 1)$$

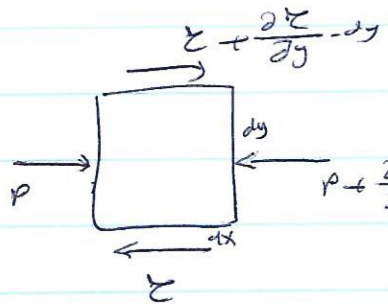
$$u = u(x,y)$$

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$$

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y}$$

The forces acting on a surface are due to pressure
viscous effects



$$F_{surface,x} = \left\{ \left[p - \left(p + \frac{\partial p}{\partial x} \cdot dx \right) \right] (dy \cdot 1) - \left[\tau - \left(\tau + \frac{\partial \tau}{\partial y} \cdot dy \right) \right] (dx \cdot 1) \right\}$$

$$= \left(\frac{\partial \tau}{\partial y} \cdot \cancel{dy} - \frac{\partial p}{\partial x} \cdot \cancel{dx} \right) (dx \cdot dy \cdot 1)$$

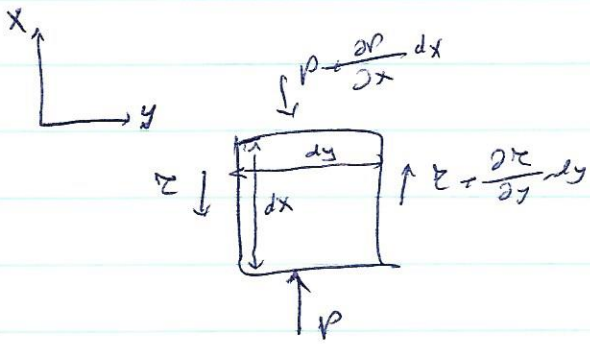
$$\tau = \mu \cdot \frac{du}{dy}$$

$$\delta m \cdot a_x = F_{surface,x} + \cancel{F_{body,x}}^{0}$$

$$\rho(\cancel{dx\, dy \cdot 1}) \cdot \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \left(\mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}\right) \cdot (\cancel{dx\, dy \cdot 1})$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}$$

In y-direction for natural convection,



$$\delta m \cdot a_x = F_x$$

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$F_x = \left\{P - \left(P + \frac{\partial P}{\partial x} dx\right)\right\}(dy \cdot 1) - \left\{\tau - \left(\tau + \frac{\partial \tau}{\partial y} dy\right)\right\}(dx \cdot 1) - \rho g\,(dx \cdot dy \cdot 1)$$

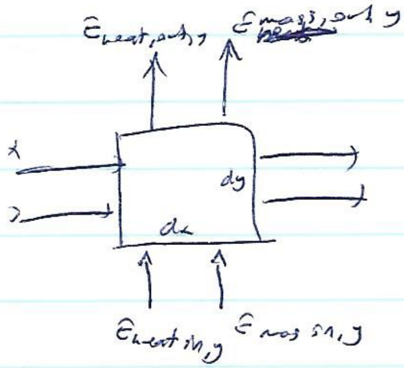
$$F_x = \frac{\partial \tau}{\partial y} dy\, dx \cdot 1 - \frac{\partial P}{\partial x} dx\, dy \cdot 1 - \rho g\,(dx\, dy \cdot 1)$$

$$\tau = \mu\frac{du}{dy} \Rightarrow$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} - \rho g$$

Conservation of Energy Equation

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ heat} + \left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ work} + \left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ mass} = 0$$



$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ mass,x} = (\dot{m}\cdot h)_x - \left((\dot{m}h)_x + \frac{\partial(\dot{m}h)}{\partial x}dx\right)$$

h, enthalpy.

$$= -\frac{\partial(\dot{m}h)}{dx}\cdot dx$$

$$= -\frac{\partial}{\partial x}\left(\rho u\cdot(dy\cdot 1)\cdot C_p T\right)\cdot dx$$

$$= -\rho C_p\cdot\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)(dx\,dy)$$

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ mass,y} = (\dot{m}h)_y - \left((\dot{m}h)_y + \frac{\partial(\dot{m}h)}{\partial y}dy\right)$$

$$= -\frac{\partial}{\partial y}(\dot{m}h)dy$$

$$= -\frac{\partial}{\partial y}\left(\rho V\cdot(dx\cdot 1)\cdot C_p T\right)$$

$$= -\rho C_p\cdot\left(V\frac{\partial T}{\partial y} + T\cdot\frac{\partial V}{\partial y}\right)\cdot(dx\,dy)$$

$$\left(\hat{E}_{in} - \hat{E}_{out}\right)_{by\ mass} = \left\{-\rho C_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right) - \rho C_p\left(V\frac{\partial T}{\partial y} + T\frac{\partial V}{\partial y}\right)\right\}dx\,dy$$

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ mass} = \left\{ -\rho c_p \left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x} + v\frac{\partial T}{\partial y} + T\frac{\partial v}{\partial y} \right) \right\} dx\,dy$$

$$= -\rho c_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + T\cdot \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)}_{\substack{\shortparallel \\ 0 \\ \text{continuity equation}}} \right) dx\,dy$$

$$\Rightarrow$$

$$\left.\begin{aligned} \left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ mass} &= -\rho c_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) dx\,dy \\ \left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ heat} &= k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dx\,dy \end{aligned}\right\} \Rightarrow$$

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ mass} + \left(\dot{E}_{in} - \dot{E}_{out}\right)_{by\ heat} = 0 \Rightarrow$$

$$\boxed{\rho c_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}$$

$$\rho c_p \left\{ \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$