exp.

Consider steady incompressible laminar flow of a film of viscous liquid falling down on a vertical wall. The film thickness is "h" and gravity acts in the negative "z" direction. There is no applied pressure driving the flow — the liquid falls by gravity alone. Calculate the velocity profile in the film! You might assume negligible shear on the outer surface of the flow. (free-surface!)

$p = p_{atm}.$

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial z} = 0.$$

$$\underbrace{\cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial v}{\partial y}}}_{\text{no-flow in } x-y \text{ directions.}}$$

$\frac{\partial w}{\partial y} = 0$ (no flow in y-)

$\frac{\partial w}{\partial x} \neq 0$

$u = v = 0$ only flow in z-direction.

$g_x = g_y = 0.$ only -z-

no-flow in x and y-directions!

$$\rho\left(\cancel{\frac{\partial w}{\partial t}} + \underset{0}{\cancel{u}}\frac{\partial w}{\partial x} + \underset{0}{\cancel{v}}\frac{\partial w}{\partial y} + w\underbrace{\frac{\partial w}{\partial z}}_{0 \text{ (continuity)}}\right) = \underbrace{-\cancel{\frac{\partial p}{\partial z}}}_{\text{no pressure gradient!}} + \underbrace{\rho g_z}_{-\rho g}$$

$$+ \mu\left(\frac{\partial^2 w}{\partial x^2} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \underbrace{\frac{\partial^2 w}{\partial z^2}}_{0}\right)$$

$$\mu\frac{\partial^2 w}{\partial x^2} - \rho g = 0$$

$$\mu\frac{d^2 w}{dx^2} = \rho g$$

$$\frac{d^2 w}{dx^2} = \frac{\rho g}{\mu}$$

$$\frac{dw}{dx} = \frac{\rho g}{\mu} \cdot x + c_1$$

B.C. no-slip.

$$w(0) = 0 \Rightarrow c_2 = 0$$

$$\left.\frac{dw}{dx}\right|_{x=h} = 0 \quad \{ \text{negligible shear!} \} \quad \left\{ \begin{array}{l} \tau = 0 \\ -\mu \frac{dw}{dx} = 0 \end{array} \right\}$$

$$\left.\frac{dw}{dx}\right|_{x=h} = \frac{\rho g}{\mu} \cdot h + c_1 = 0$$

$$c_1 = -\frac{\rho g}{\mu} \cdot h$$

$$w(x) = \frac{\rho g}{\mu} \frac{x^2}{2} - \frac{\rho g h}{\mu} x$$

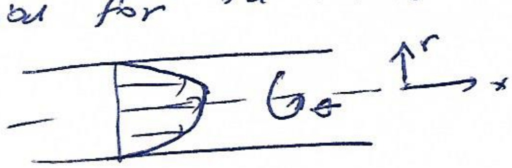
$$w(x) = \frac{\rho g x}{2\mu} (x - 2h)$$

Since $x < h$ in the film,

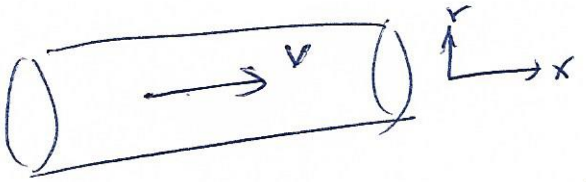
w is "−" as expected!

ex

Consider steady, incompressible laminar parallel flow of a Newtonian fluid in a pipe where $u = 0/z$ ignoring the effects of gravity. If a constant pressure gradient is applied in the x-direction, derive an expression for the velocity inside the pipe.



continuity: $\frac{1}{r}\cdot\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\cdot\frac{\partial (u_\theta)}{\partial \theta} + \frac{\partial u_x}{\partial x} = 0$



$\frac{\partial p}{\partial x} = \frac{P_2 - P_1}{x_2 - x_1}$

parallel flow $u_r = 0$ ✓

no-swirl $u_\theta = 0$ ✓ $\Big\}$ $\Rightarrow$ $\frac{\partial u_x}{\partial x} = 0$

u is not a function of x!

axi-symmetric in θ!

so, $u = f(r)$.

x-direction

$$\rho\left(\underbrace{\cancel{\frac{\partial u_x}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{u_r\frac{\partial u_x}{\partial r}}}_{u_r=0} + \cancel{\frac{u_\theta}{r}\cdot\frac{\partial u_x}{\partial \theta}} + \cancel{\;\;} + u_x \underbrace{\cancel{\frac{\partial u_x}{\partial x}}}_{0}\right) = -\frac{\partial p}{\partial x} + \cancel{\rho g_x} + \mu\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) + \cancel{\frac{1}{r^2}\cdot\frac{\partial^2 u_x}{\partial \theta^2}} + \underbrace{\cancel{\frac{\partial^2 u_x}{\partial x^2}}}_{\frac{\partial u_x}{\partial x}=0}\right]$$

$$-\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)\right) = 0$$

$u = f(r)$

$$\frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\cdot\frac{dP}{dx}$$

$$r\frac{du}{dr} = \frac{r^2}{2\mu}\cdot\frac{dp}{dx} + C_1$$

$$u(r) = \frac{r^2}{4\mu}\cdot\frac{dp}{dx} + C_1 \ln(r) + C_2$$

At $r = 0$ $\frac{\partial u}{\partial r} = 0$ axi-symmetry! $\Rightarrow$

$$\underline{\underline{C_1 = 0}}$$

$r = R$ $u(r=R) = 0$

$$\frac{R^2}{4\mu}\cdot\frac{dp}{dx} + C_2 = 0 \Rightarrow$$

$$C_2 = -\frac{R^2}{4\mu}\cdot\frac{dp}{dx}$$

$$u(r) = \frac{r^2}{4\mu}\cdot\frac{dp}{dx} + \frac{R^2}{4\mu}\frac{dp}{dx}$$

$$u(r) = \frac{1}{4\mu}\cdot\frac{dp}{dx}(r^2 - R^2)$$ ✓