Laplace Transforms

LAPLACE TRANSFORMS

- In Food Engineering we develop a number of mathematical models that describe the dynamic operation of selected processes.

- Solving such models requires either analytical or numerical integration of the differential equations.

- Laplace transformation is a mathematical tool that **converts differential equations to algebraic equations.**

- It reduces the effort required to solve the model.

Definition

The laplace transform of a function f(t) is defined as:

$$f(s) = \int_0^\infty f(t) e^{-st} dt$$

- f(t) is a function depends on time
- f(s) function is the Laplace Transform of f(t)
- Laplace transformations can represented by different symbols.

$$f(s) = L\{f(t)\} = F(s) = \overline{f(s)} = Y(s)$$

Let’s apply Laplace transform to f(t) = 1

$$f(s) = \int_0^\infty (1) e^{-st} dt = -\left.\frac{e^{-st}}{s}\right|_0^\infty = -0 + \frac{e^{-s*0}}{s} = \frac{1}{s}$$

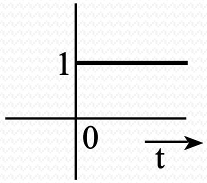
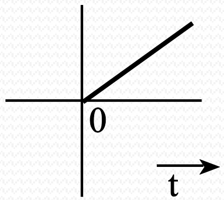
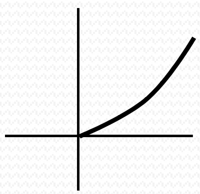
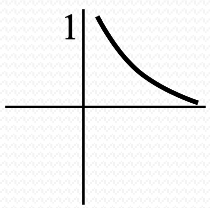
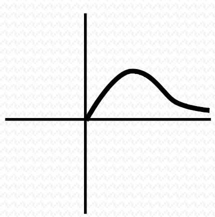
$$\boxed{L\{1\} = \frac{1}{s}}$$

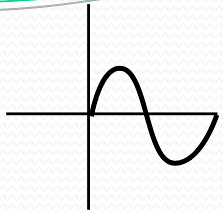
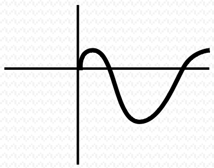
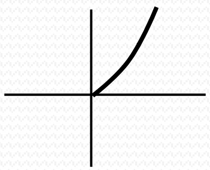
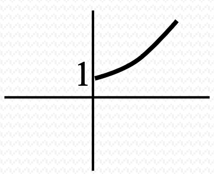
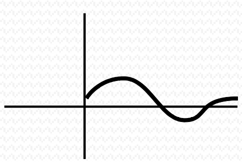
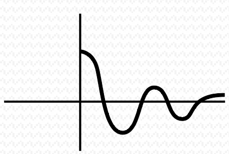
Therefore the laplace transform of f(t)=1 is:

$$f\{s\} = \frac{1}{s}$$

- In solving differential equations:
  - First apply Laplace transformation
  - Then solve the equations algebraically to obtain Y(s)
  - Finally take the inverse Laplace transform

- Laplace transform tables helps us to convert the functions.

FUNCTION	GRAPH	TRANSFORM
1		$1/s$
t		$1/s^2$
t^n		$\frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, \dots$
e^{-at}		$\frac{1}{s+a}$
$t^n . e^{-at}$		$\frac{n!}{(s+a)^{n+1}}$

FUNCTION	GRAPH	TRANSFORM
$\sin kt$		$\frac{k}{s^2 - k^2}$
$\cos kt$		$\frac{s}{s^2 - k^2}$
$\sinh kt$		$\frac{k}{s^2 + k^2}$
$\cosh kt$		$\frac{s}{s^2 + k^2}$
$e^{-at} \sin kt$		$\frac{k}{(s+a)^2 + k^2}$
$e^{-at} \cos kt$		$\frac{s+a}{(s+a)^2 + k^2}$

PROPERTIES OF LAPLACE TRANSFORM

1. Time variable is between zero and infinity($0 < t < \infty$)

2. Laplace transformation is always applied to linear differential equations

3. The laplace transform of the sum of two function is equal to laplace transform of individual functions separately.

$$L\{a f_1(t) + b f_2(t)\} = aL\{f_1(t)\} + bL\{f_2(t)\}$$

a and b are constants, f_1 and f_2 are the functions.

4. Variable *t* is eliminated by variable *s* during laplace transformation.

Laplace Transform of a Derivative

$$£\{ f'(t)\} = sf(s) - f(0)$$

$$£\{ f''(t)\} = s^2 f(s) - sf(0) - f'(0)$$

$$£\left\{ \frac{d^n f(t)}{dt^n} \right\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) \dots - f^{n-1}(0)$$

Laplace Transform of an Integral

$$£\left\{ \int_0^t f(t)\,dt \right\} = \frac{f(s)}{s}$$

How to solve differential equ.?

$y = f(t) \quad y(0) = 1 \quad L(y) = Y(s)$

- y → is a function of t
- y= 1 @ t=0
- Laplace transform of function *y* is equal to **Y(s)**

$$5\frac{dy}{dt} + 4y = 2$$

- Solve the differential equation above using laplace transforms.

How to solve differential equ.?

$$y = f(t) \quad y(0) = 1 \quad L(y) = Y(s)$$

$$5\frac{dy}{dt} + 4y = 2$$

- Apply laplace transform to each term
- Solve the equations algebraically to obtain Y(s)
- Apply inverse transform to each term
- You may use the Laplace table.

Example

Solve the differential equation by Laplace.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 1$$

$$y(0) = y'(0) = 0$$

Final and Initial Value Theorem

Final and initial value theorem gives information about the process in process control and process dynamics.

$$\lim_{t\to\infty}\left[f(t)\right] = \lim_{s\to 0}\left[s\,F(s)\right]$$

FINAL VALUE THEOREM

- It is used to find the steady state value of the function (t=∞)

$$\lim_{t\to 0}\left[f(t)\right] = \lim_{s\to \infty}\left[s\,F(s)\right]$$

INITIAL VALUE THEOREM

- It is used to find the initial value of the function at t=0

Examples of Final and Initial Value Theorem

$$Y(s) = \frac{5s+2}{s(5s+4)}$$

- Laplace of a function

$$sY(s) = \frac{5s+2}{(5s+4)}$$

$$\lim_{s\to\infty} sY(s) = \frac{5s+2}{5s+4} = 1$$

- Initial value theorem

$$\lim_{s\to 0} sY(s) = \frac{5s+2}{5s+4} = 0.5$$

- Final value theorem

EXAMPLE

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2 \quad y(0) = y'(0) = 0$$

- ODE
- Apply laplace transform to each term
- Solve for Y(s)
- Apply the method of partial fractions
- Determine the inverse transform

SOLUTION

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2 \quad y(0) = y'(0) = 0$$

$$s^2Y(s) + 6sY(s) + 8Y(s) = 2/s$$

$$Y(s) = \frac{2}{s(s+2)(s+4)}$$

$$Y(s) = \frac{1}{4s} + \frac{-1}{2(s+2)} + \frac{1}{4(s+4)}$$

$$y(t) = \frac{1}{4} - \frac{e^{-2t}}{2} + \frac{e^{-4t}}{4}$$

- ODE
- Apply laplace transform to each term
- Solve for Y(s)
- Apply the method of partial fractions
- Determine the inverse transform