MATHEMATICAL MODELING

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A model is a mathematical abstraction of a real process.

The relation between input and output variables can also be defined as Model.

Processes can be expressed by differential equations.

◆ *Steady State*

★ A system in a steady state has numerous properties (T, P, C, V, h, ...) that are unchanging in time

(Food processes are mostly operated at steady state)

◆ Unsteady State

★ Properties of the system changes with time

- ***Start-up***
- ***Shut-down***
- ***Batch***

◆ *Process Dynamics*

★ Response of the process to a change in input variables.
Behavior of the process with time
Response depends on ;

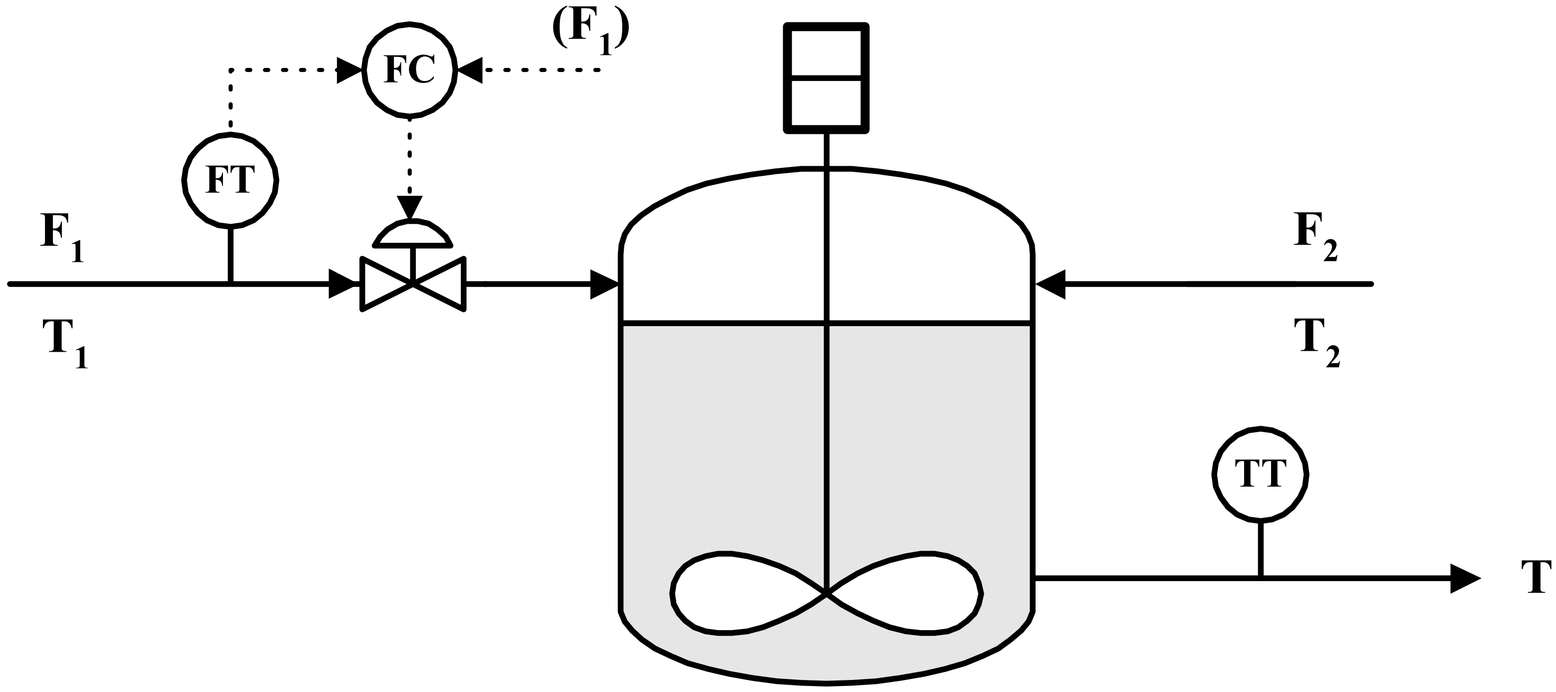
- **Properties of the input**
- **Characteristics of the process**

Classification of the Models

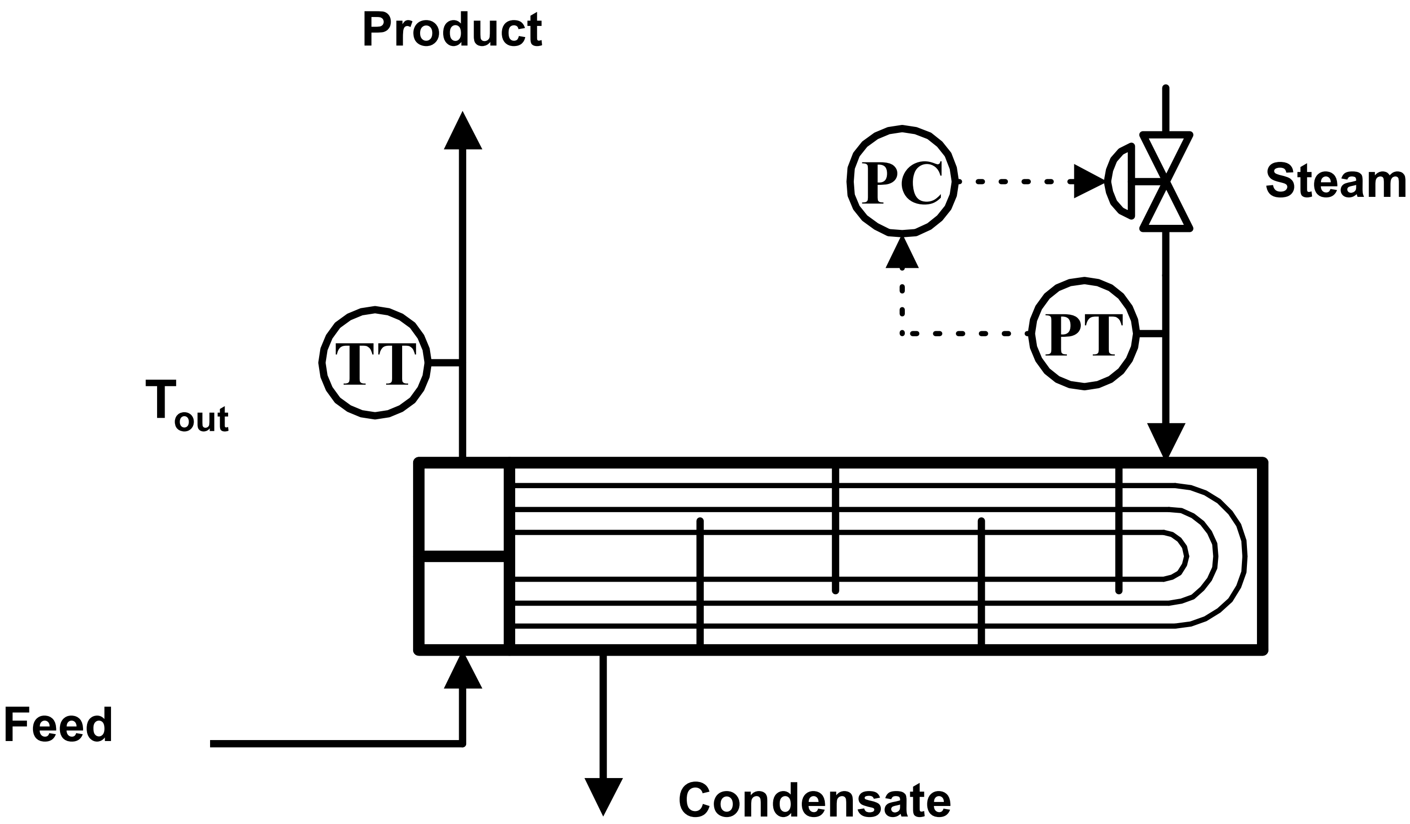
- **Lumped parameter models**- properties do not change with position within the system
  - Example: *a well mixed tank*
  - Macroscopic (total) balance is needed.

- **Distributed parameter models**- properties change with position
  - Microscopic (componenet) balance is needed.

Lumped parameter process



Distributed parameter process

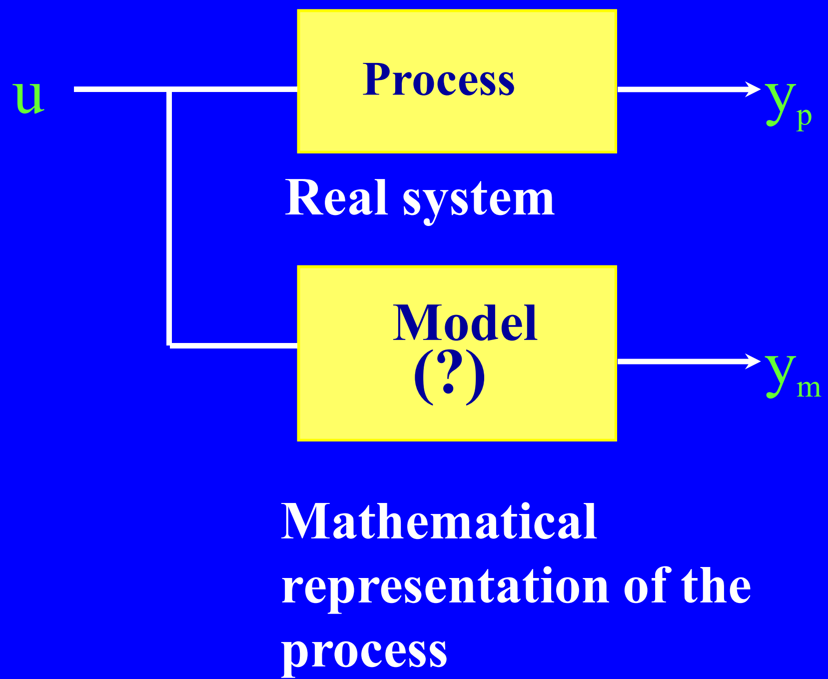


Mathematical Modeling

- Model needs to be the best approximation of the true process.

In theory:

- $y_m = y_p$



In Mathematical Modeling, Conservation Laws are mainly used.

- ## Mass Balance

Rate of mass accum. = (Rate of Mass Input) - (Rate of Mass Output) ± (Rate of Mass Generated/Consumed)

- ## Energy Balance

Rate of energy accum.= (Rate of Energy Input) - (Rate of Energy Output) ± (Rate of Energy Generated/Consumed)

- ## Momentum Balance

Rate of momentum accum. = (Rate of Momentum Input) - (Rate of Momentum Output) + Sum of net forces acting on the element

In addition; Thermodynamic Laws, Newton's Law of Viscosity, Fourier's Law, Fick's Law etc......

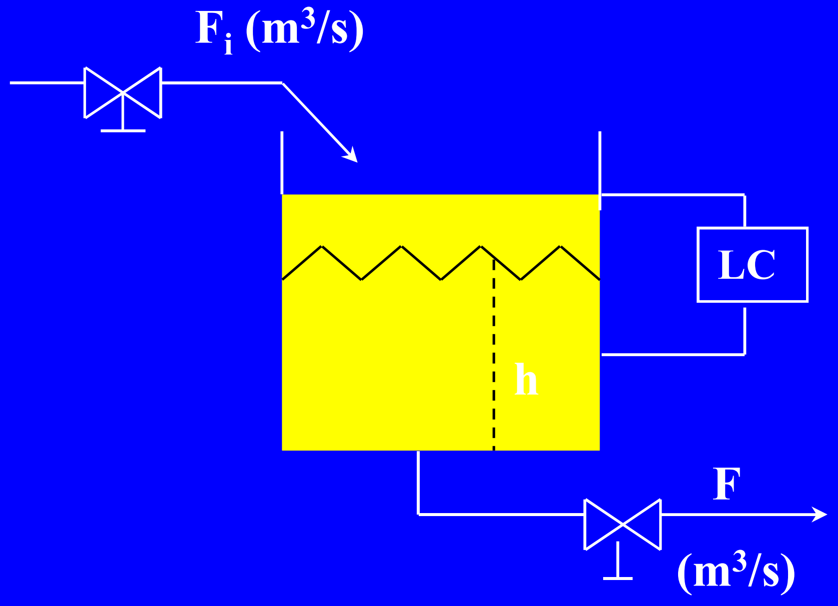
Example 1: Liquid Storage System

Problem: Control the liquid level in the tank.

Controlled variable: h

Input variables: F_i, F

Manipulated variable: ?



Control system can be configured in different ways.

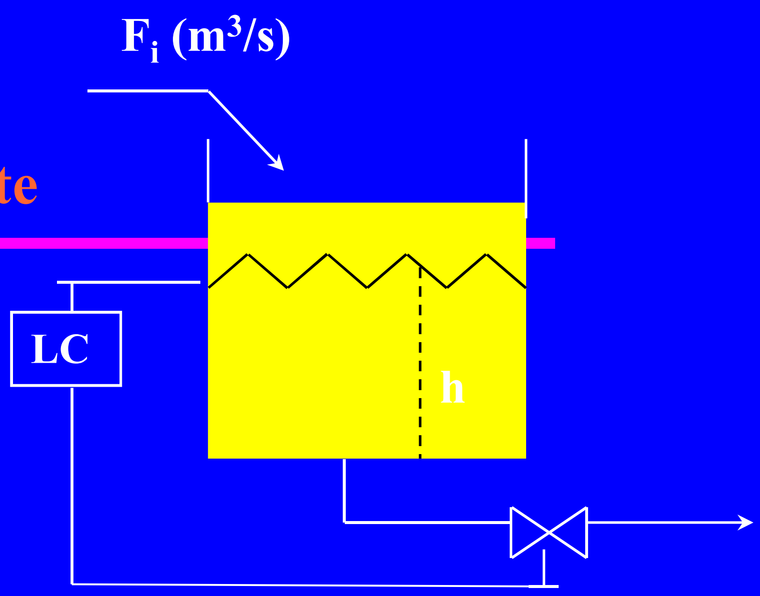
Process model will be different depending on your control system.

➔Tank cross sectional area

A_c= constant

1. Configuration:

Manipulated variable: Outlet flow rate



Mass balance:

➔ For unsteady state

Rate of mass accum. = (Rate of Mass Input) - (Rate of Mass Output)

$$\frac{d(\rho A_c h)}{dt} = \rho F_i - \rho F \qquad \rho \text{ , } A_c \text{ constant}$$

$$A_c \frac{dh}{dt} = F_i - F \qquad \text{.........................} \quad (1)$$

➔ For steady state

$$0 = F_{is} - F_s \qquad \text{.........................} \quad (2)$$

(s : represents the steady state value)

Subtract Equation (2) from Equation (1);

➔ Because we want to express the variables in terms of deviation variables

$$A_c \frac{dh}{dt} = (F_i - F_{is}) - (F - F_s) \qquad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

Deviation Variables:

If we define deviation variables as

- $y = h - h_s$
- $d = F_i - F_{is}$
- $u = F - F_s$

➔ Deviation variable: It shows how much a variable deviates from its initial steady state conditions.

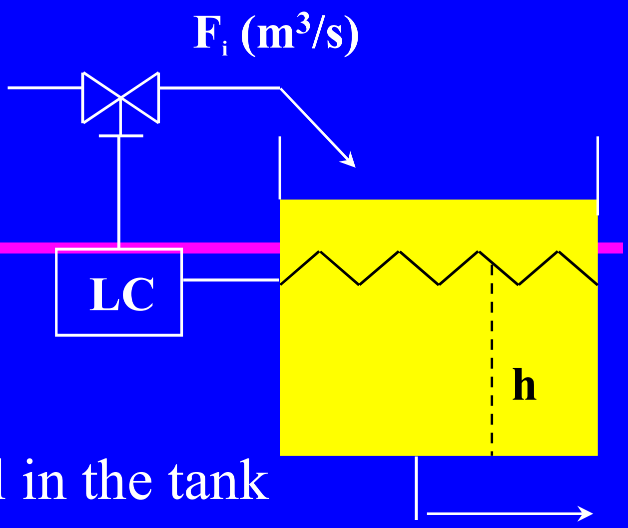
Then, the model of the process :

$$\frac{dy}{dt} = \frac{1}{A_c} d - \frac{1}{A_c} u$$

This equation shows how the controlled variable (y) depends on disturbances (d) and manipulated variable(u).

2. Configuration:

Manipulated variable: Input flow rate



$$A_c \frac{dh}{dt} = F_i - F$$

In this case F is related to the liquid level in the tank

Assume $F = ch$

$$A_c \frac{dh}{dt} = F_i - ch \quad \ldots\ldots\ldots\ldots\ldots \quad (4) \text{ (Unsteady state)}$$

$$0 = F_{is} - ch_s \quad \ldots\ldots\ldots\ldots\ldots \quad (5) \text{ (Steady state)}$$

Subtract Equation (5) from Equation (4);

$$A_c \frac{dh}{dt} = (F_i - F_{is}) - c\,(h - h_s) \;\ldots\ldots \quad (6)$$

The model of the process in terms of deviation variables:

$y = h - h_s$, $u = F_i - F_{is}$

$$\frac{dy}{dt} + \frac{c}{A_c} y = \frac{1}{A_c} u$$

This model shows how the controlled variable (y) depends on manipulated variable(u). Not a good model.

In reality; $F = c\, h^{1/2}$

Not linear!!

We need to linearize because laplace can only be applied to linear dif. equ.

$$A_c \frac{dh}{dt} = F_i - F$$

$$A_c \frac{dh}{dt} = F_i - c\sqrt{h}$$

Not linear