MATHEMATICAL MODELING

LINEARIZATION

- **Each term in the model is linearized around the operating point (steady state value of the process).**
  - ➔ **When linearized, the model would be valid only around the operating point.**

Taylor Series Expansion:

For a function that has one variable (linearization around x_s)

$$F(x) = F(x_s) + \left.\frac{dF}{dx}\right|_{x_s} (x - x_s) + \frac{1}{2!}\left.\frac{d^2F}{dx^2}\right|_{x_s} (x - x_s)^2 + R$$

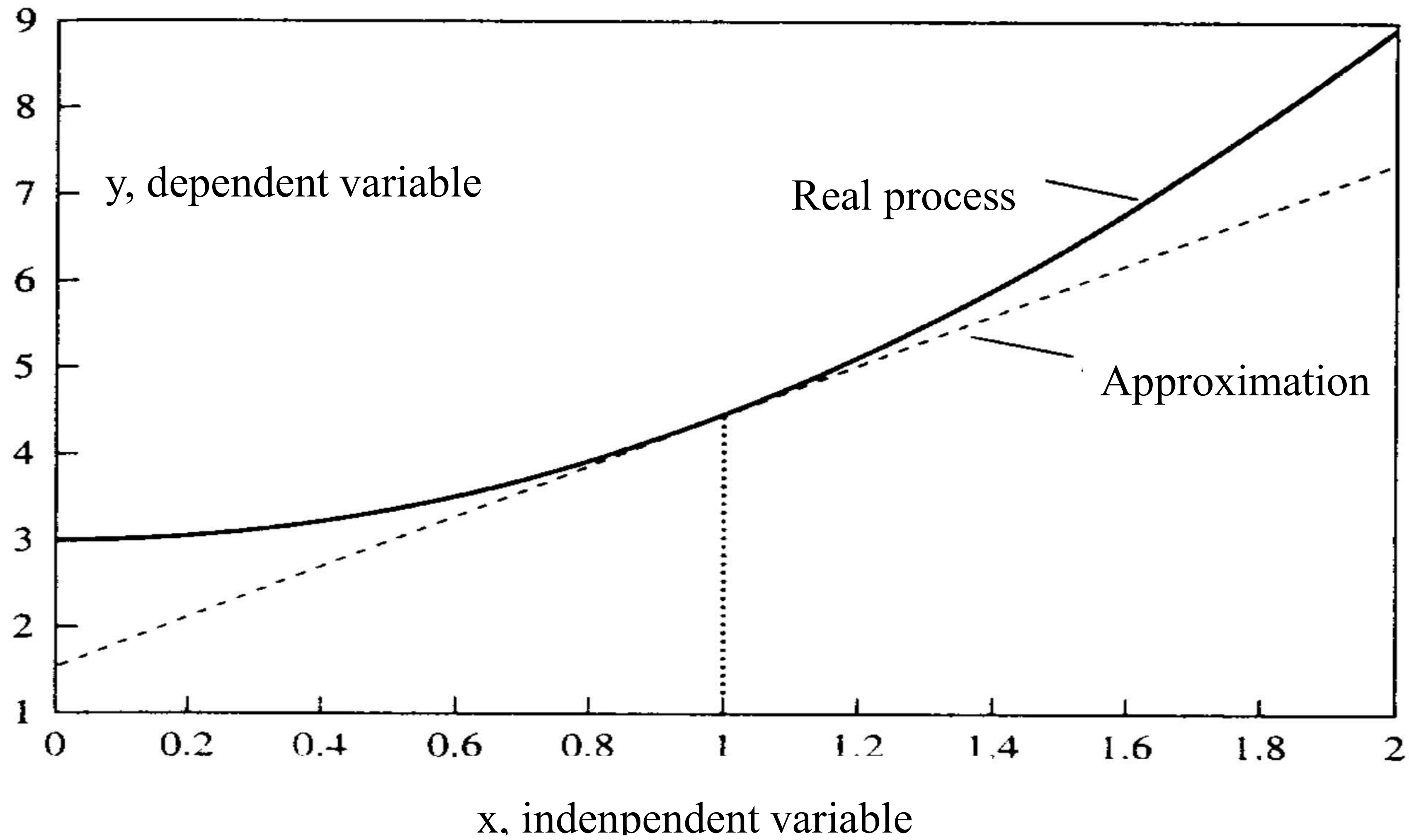
For a function that has two variables (linearization around x_s)

$$F(x_1, x_2) = F(x_{1s}, x_{2s}) + \left.\frac{dF}{dx_1}\right|_{x_{1s}, x_{2s}} (x_1 - x_{1s}) + \left.\frac{dF}{dx_2}\right|_{x_{1s}, x_{2s}} (x_2 - x_{2s})$$

$$+ \frac{1}{2!}\left.\frac{d^2F}{dx_1^2}\right|_{x_{1s}, x_{2s}} (x_1 - x_{1s})^2 + \frac{1}{2!}\left.\frac{d^2F}{dx_2^2}\right|_{x_{1s}, x_{2s}} (x_2 - x_{2s})^2$$

$$+ \frac{d^2F}{dx_1\, dx_2} (x_1 - x_{1s})(x_2 - x_{2s}) + R$$

Linearization of a non-linear function around $x_s = 1$



Linearization of $y=1.5x^2 + 3$ at $x_s = 1$

Linearization example:

$$F(x) = x^{1/2} \qquad F(x) = x_s^{1/2} + \frac{1}{2} x_s^{-1/2} \left(x - x_s\right)$$

Let's go back to our liquid level control problem

$$A_c \frac{dh}{dt} = F_i - F$$

$$A_c \frac{dh}{dt} = F_i - c\sqrt{h}$$

→ Nonlinear

Let's linearize around a steady state value, h_s :

$$c\sqrt{h} \approx c\sqrt{h_s} + \left.\frac{d(ch^{1/2})}{dh}\right|_{h = h_s} (h - h_s)$$

$$c\sqrt{h} \approx c\sqrt{h_s} + \frac{c}{2\sqrt{h_s}} (h - h_s)$$

Insert it in the original equ.;

$$A_c \frac{dh}{dt} = F_i - c\sqrt{h_s} - \frac{c}{2\sqrt{h_s}} (h - h_s)$$

Steady state: $0 = F_{is} - c\sqrt{h_s}$

($h=h_s$, $F_i = F_s$)

Subtract the steady state equation from unsteady state equation:

$$A_c \frac{dh}{dt} = (F_i - F_{is}) - \frac{c}{2\sqrt{h_s}} (h - h_s)$$

Define deviation variables:

$$y = h - h_s$$

$$u = F_i - F_{is}$$

$$A_c \frac{dy}{dt} = u - \frac{c}{2\sqrt{h_s}} y$$

$$A_c \frac{dy}{dt} + \frac{c}{2\sqrt{h_s}} y = u$$

A more realistic model

Let's solve the differential equ. and find Transfer function of the process

$$L\left[\frac{dy}{dt} + \frac{c}{Ac}\, y\right] = L\left[\frac{1}{Ac}\, u\right]$$

(2. configuration)

$$s\,y(s) + \frac{c}{Ac}\, y(s) = \frac{1}{Ac}\, u(s)$$

Multiply with Ac and divide by c

$$\underbrace{\left(\frac{Ac}{c}\right)}_{\tau} s\,y(s) + y(s) = \frac{1}{c}\, u(s)$$

$$(\tau s + 1)\, y(s) = \frac{1}{c}\, u(s)$$

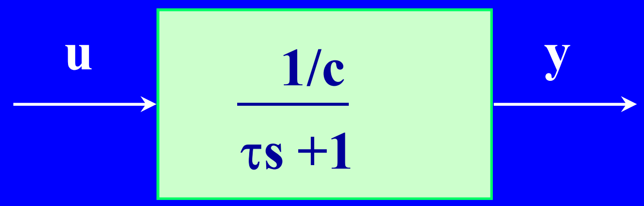
$$\frac{y(s)}{u(s)} = \frac{1/c}{\tau s + 1}$$

Let's find the transfer function of the process

- **The ratio of laplace of the output variable to laplace of the input variable**

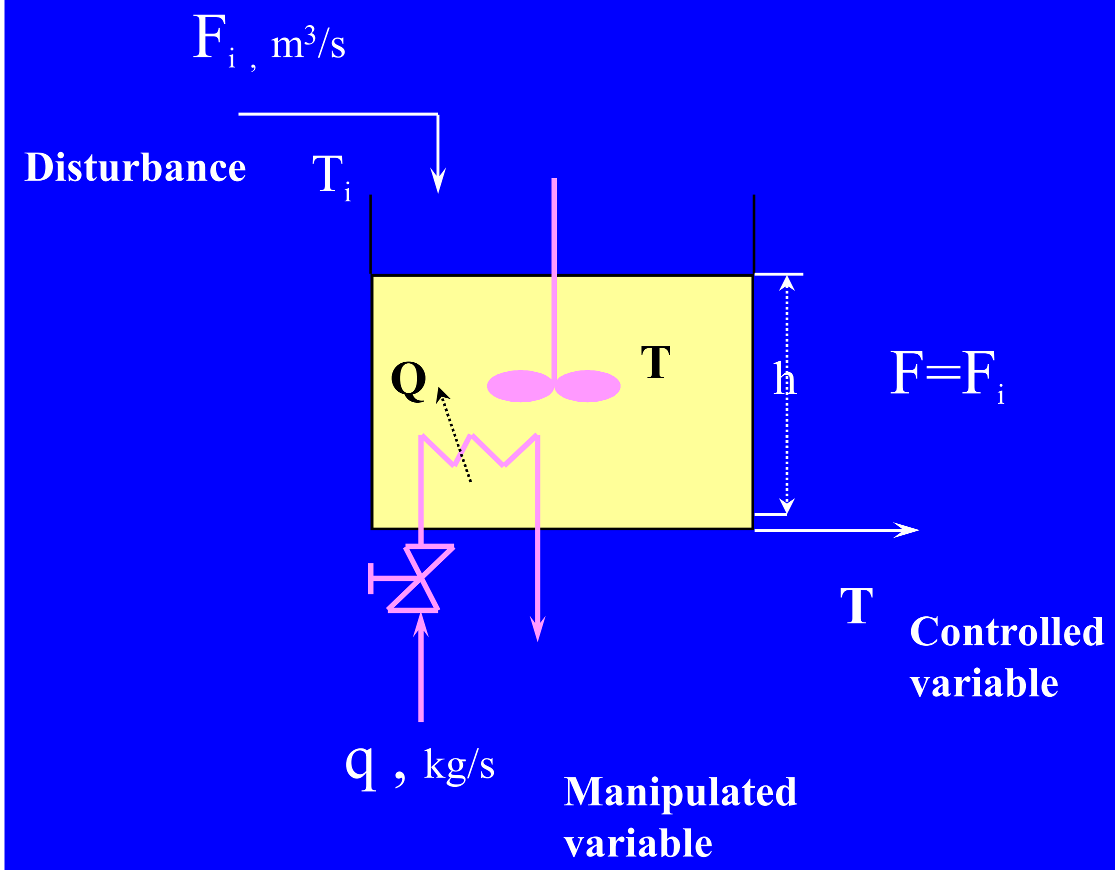
$$G(s) = \frac{y(s)}{u(s)} = \frac{1/c}{\tau s + 1}$$

(For liquid level control system)



Process

Example 2: Stirred-tank heating system



Input Variables

q : Manipulated variable(q_{steam})

T_i : Disturbance

Output Variables

T : Controlled variable

$$\frac{dy}{dt} + \frac{F}{V} y = \frac{\lambda}{\rho V C_p} u + \frac{F}{V} d$$

Deviation var.: $y = T - T_s$, $u = q - q_s$, $d = T_i - T_{is}$

Take laplace:

$$s\, y(s) + \frac{F}{V} y(s) = \frac{\lambda}{\rho V C_p} u(s) + \frac{F}{V} d(s)$$

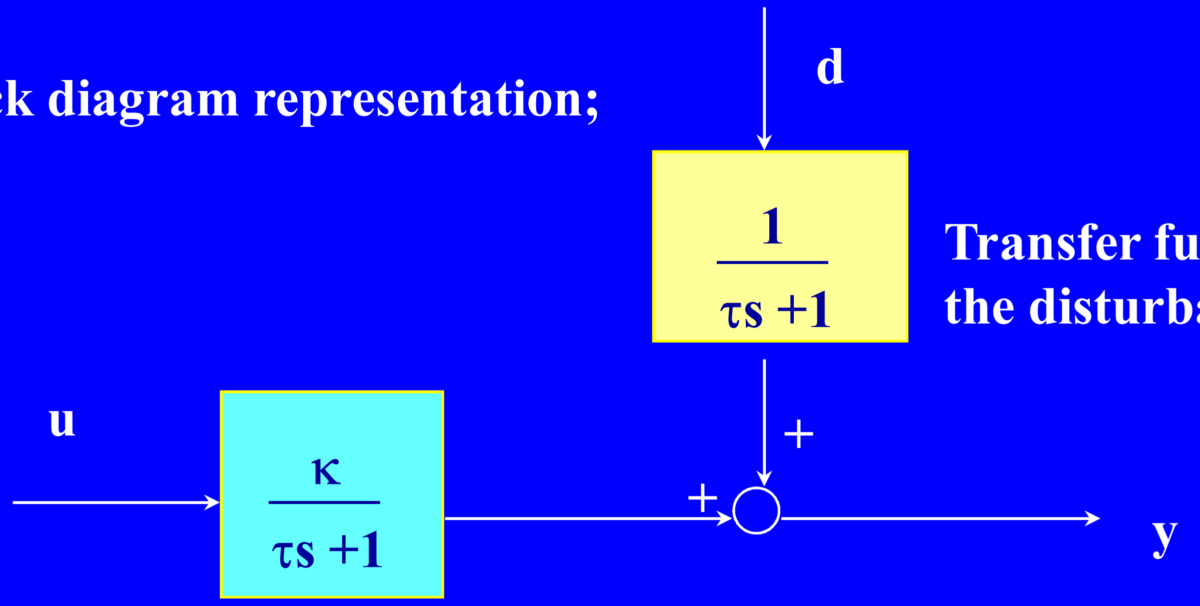
(Multiply each term with V/F)

$$\left(\frac{V}{F} s + 1\right) y(s) = \frac{\lambda}{\rho F C_p} u(s) + d(s)$$

$$y(s) = \frac{K}{\tau s + 1} u(s) + \frac{1}{\tau s + 1} d(s)$$

$$\tau = \frac{V}{F} \quad , \quad \kappa = \frac{\lambda}{\rho\, F\, C_p}$$

Block diagram representation;



Transfer function of the disturbance

Transfer function of the process