Integral form of the fundamental postulates

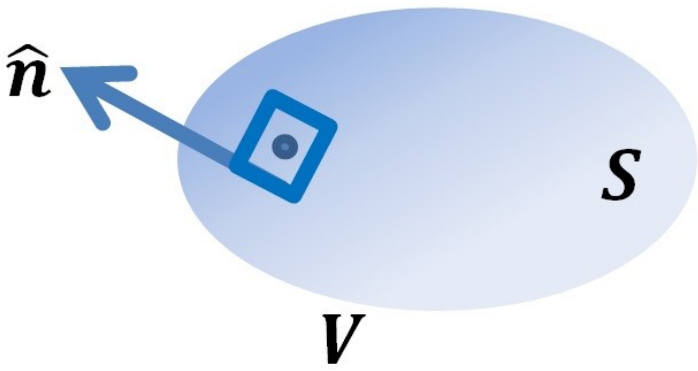
$$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}$$ Gauss' Law in differential form

$\Rightarrow$

$$\int_V \nabla \cdot \vec{E} dV = \int_V \frac{\rho_v}{\epsilon_0} dV = \frac{1}{\epsilon_0}\int_V \rho_v dV$$

$$= \frac{Q_{enclosed}}{\epsilon_0}$$

By Divergence Theorem,



$$\int_V \nabla \cdot \vec{E} dV = \oint_S \vec{E} \cdot \overrightarrow{dS}$$

$\Rightarrow$

$$\oint_S \vec{E} \cdot \overrightarrow{dS} = \frac{Q_{enclosed}}{\epsilon_0}$$ Gauss' Law in integral form

&

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}$$ Gauss' Law in differential form

b)

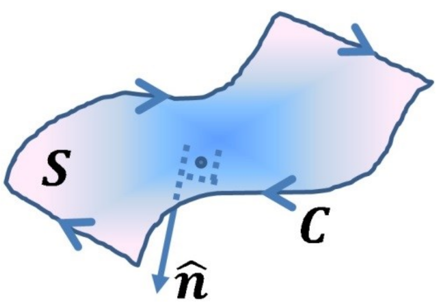
$$\nabla \times \vec{E} = \vec{0}$$ 2nd Postulate in differential form

$\Rightarrow$

$$\int_S \nabla \times \vec{E} \cdot \overrightarrow{dS} = \oint_S \vec{0} \cdot \overrightarrow{dS} = 0$$

By Stoke's Theorem,

$$\int_S \nabla \times \vec{E} \cdot \overrightarrow{dS} = \oint_C \vec{E} \cdot \overrightarrow{dl}$$



⇒

$$\oint_C \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dl}} = 0$$ 2nd Postulate in integral form

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} = \vec{0}$$ 2nd Postulate in differential form

Note:

- $\int_C \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dl}}$ gives the voltage drop along path C
- $\oint_C \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dl}}$ which is the algebraic sum of the voltages drops around any closed loop (circuit) add up to zero
- i.e.,

$$\oint_C \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dl}} = 0$$ **Kirchoff's Voltage Law**

- Since $\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} = \vec{0}$, line integral of $\vec{E}$ depends only on the end points

⇒ Electrical work done for a moving charge around a closed path in an electrostatic field is zero

⇒ $\oint_C \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dl}} = 0$ states conservation of energy in an electrostatic field

Gauss' Law

Consider a charge distribution in free space:

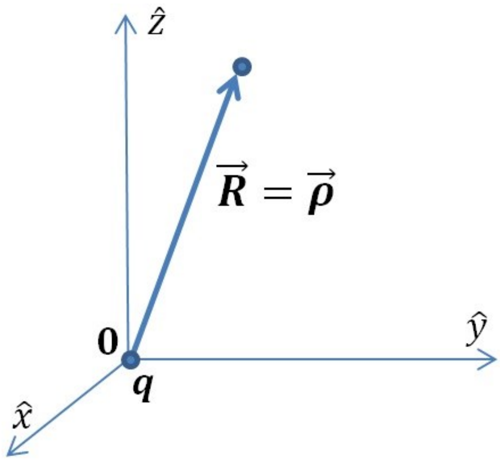
Discrete Charge Distribution	Volume charge density	Surface charge density	Line charge density
q_i	ρ_v	ρ_s	ρ_v
$Coulomb$	$\frac{Coulomb}{m^3}$	$\frac{Coulomb}{m^2}$	$\frac{Coulomb}{m}$

These charges create an electric field, $\vec{\boldsymbol{E}}$. If $\boldsymbol{S}$ is a closed surface,

$$\oint_S \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dS}} = \frac{Q_{enclosed}}{\epsilon_0}$$

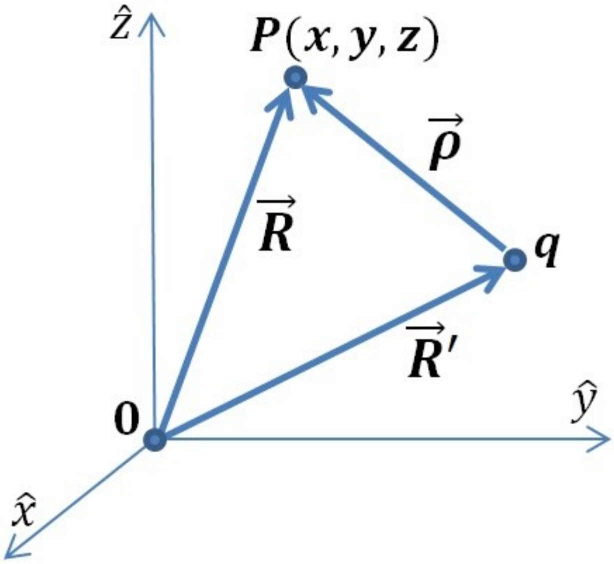
where $Q_{enclosed}$ is the total charge enclosed within surface S

Electric field vector due to the unit charge at origin:



$$\Rightarrow \qquad \vec{E} = E\hat{\rho} = \frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2} \qquad \frac{V}{m}$$

Electric field vector due to the unit charge at a point different than origin:



$\vec{R}$: *Vector from origin to the observation point P*

$\vec{R}'$: *Vector from origin to the source point*

$$\Rightarrow \qquad \vec{E} = E\hat{\rho} = \frac{q}{4\pi\epsilon_0}\frac{\vec{R} - \vec{R}'}{\left|\vec{R} - \vec{R}'\right|^3} \qquad \frac{V}{m}$$

$$\vec{\rho} = \vec{R} - \vec{R}'$$

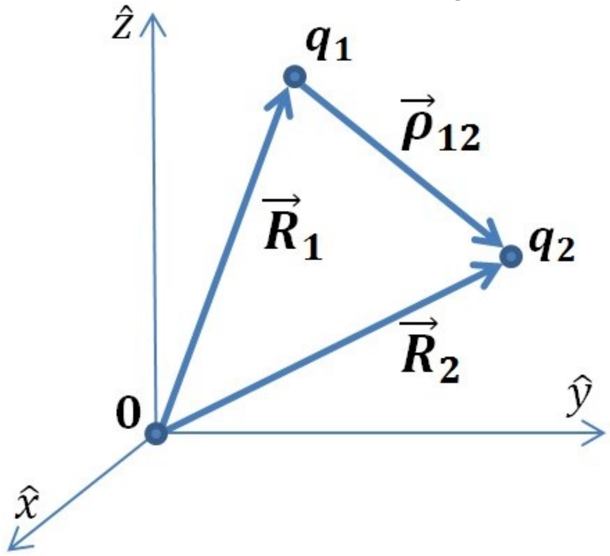
$$\hat{\boldsymbol{\rho}} = \frac{\vec{\boldsymbol{\rho}}}{\boldsymbol{\rho}}$$

$$\vec{\boldsymbol{E}} = \boldsymbol{E}\hat{\boldsymbol{\rho}} = \frac{\boldsymbol{q}}{4\pi\epsilon_0\left|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'\right|^2} \frac{\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'}{\left|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'\right|}$$

$$\vec{\boldsymbol{E}} = \boldsymbol{E}\hat{\boldsymbol{\rho}} = \frac{\boldsymbol{q}}{4\pi\epsilon_0\boldsymbol{\rho}^2}\hat{\boldsymbol{\rho}} = \frac{\boldsymbol{q}}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\boldsymbol{\rho}^2}$$

Statement of Coulomb's law:

The force between two point charges is proportional to the product of the charges & inversely proportional to the square of the distance between the charges:



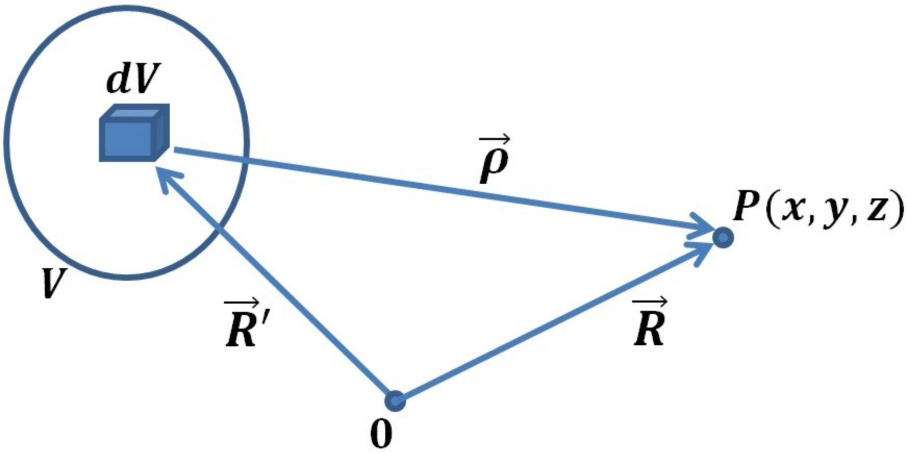
$$\vec{\boldsymbol{\rho}}_{12} = \vec{\boldsymbol{R}}_2 - \vec{\boldsymbol{R}}_1$$

$$\vec{\boldsymbol{F}}_{12} = \boldsymbol{q}_2\vec{\boldsymbol{E}}_{12}$$

$$\vec{\boldsymbol{F}}_{12} = \boldsymbol{q}_2\left(\frac{\boldsymbol{q}_1}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{\rho}}_{12}}{|\vec{\boldsymbol{\rho}}_{12}|^3}\right) = \boldsymbol{q}_2\left(\frac{\boldsymbol{q}_1}{4\pi\epsilon_0}\frac{\mathbf{1}}{|\vec{\boldsymbol{\rho}}_{12}|^2}\frac{\vec{\boldsymbol{\rho}}_{12}}{|\vec{\boldsymbol{\rho}}_{12}|}\right)$$

$$\vec{\boldsymbol{F}}_{12} = \boldsymbol{q}_2\left(\frac{\boldsymbol{q}_1}{4\pi\epsilon_0}\frac{\mathbf{1}}{|\vec{\boldsymbol{\rho}}_{12}|^2}\hat{\boldsymbol{\rho}}_{12}\right)$$

Electrostatic Potential



$\vec{R}$: *Vector from origin to the observation point P*

$\vec{R}'$: *Vector from origin to the source point*

$$\vec{\rho} = \vec{R} - \vec{R}'$$

$$\hat{\rho} = \frac{\vec{\rho}}{\rho} = \frac{\vec{R} - \vec{R}'}{|\vec{R} - \vec{R}'|}$$

$$\frac{\vec{R} - \vec{R}'}{|\vec{R} - \vec{R}'|^3} = \frac{\vec{R} - \vec{R}'}{|\vec{R} - \vec{R}'|^1} \frac{1}{|\vec{R} - \vec{R}'|^2} = \frac{\hat{\rho}}{1}\frac{1}{\rho^2} = \frac{\hat{\rho}}{\rho^2}$$

$$\frac{\vec{R} - \vec{R}'}{|\vec{R} - \vec{R}'|^3} = \frac{\hat{\rho}}{\rho^2} = \frac{\rho\hat{\rho}}{\rho^3} = \frac{\vec{\rho}}{\rho^3}$$

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int_{D'} dq' \frac{\vec{R} - \vec{R}'}{|\vec{R} - \vec{R}'|^3}$$

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int_{D'} dq' \frac{\vec{\rho}}{\rho^3}$$

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int_{D'} dq' \frac{\hat{\rho}}{\rho^2}$$

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int_{D'} \rho'_V dV' \frac{\hat{\rho}}{\rho^2}$$

$$\nabla\left(\frac{1}{R}\right) = -\frac{\widehat{R}}{R^2}$$

$$\nabla\left(\frac{1}{\left|\vec{R}-\vec{R}'\right|}\right) = -\frac{\vec{R}-\vec{R}'}{\left|\vec{R}-\vec{R}'\right|^3}$$

$$\nabla'\left(\frac{1}{\left|\vec{R}-\vec{R}'\right|}\right) = +\frac{\vec{R}-\vec{R}'}{\left|\vec{R}-\vec{R}'\right|^3} = -\nabla\left(\frac{1}{\left|\vec{R}-\vec{R}'\right|}\right)$$

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_{D'} dq'. -\nabla\left(\frac{1}{\left|\vec{R}-\vec{R}'\right|}\right)$$

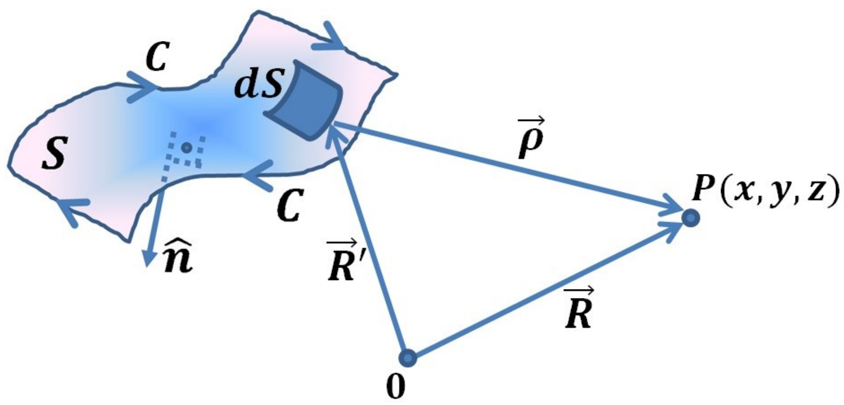
$$\vec{E}(P) = -\nabla\left(\frac{1}{4\pi\epsilon_0}\int_{D'} \frac{dq'}{\left|\vec{R}-\vec{R}'\right|}\right)$$

$$\vec{E}(P) = -\nabla(V)$$

$$V(P) = \frac{1}{4\pi\epsilon_0}\int_{D'} \frac{dq'}{\left|\vec{R}-\vec{R}'\right|}$$

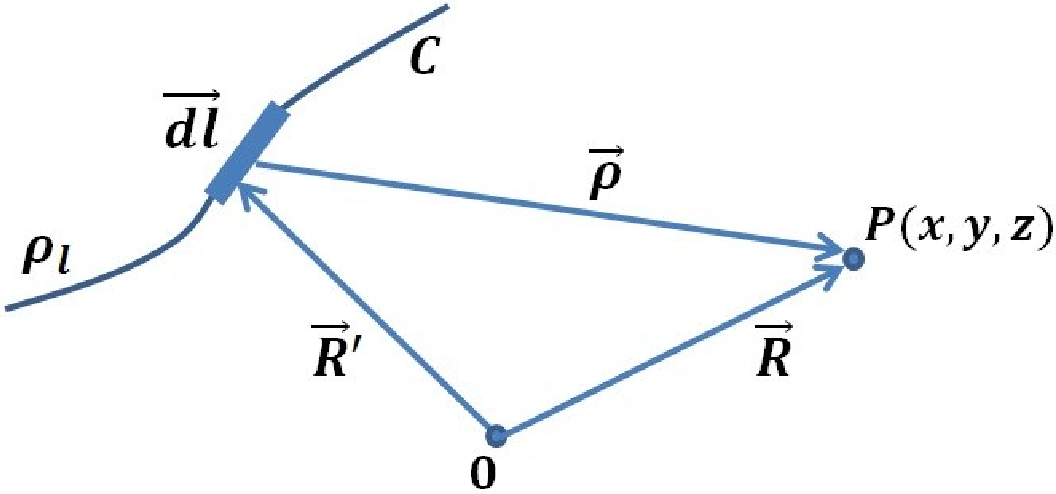
Check $\nabla \times \vec{E} = \vec{0}$

Check $\nabla \times (-\nabla V) = 0$



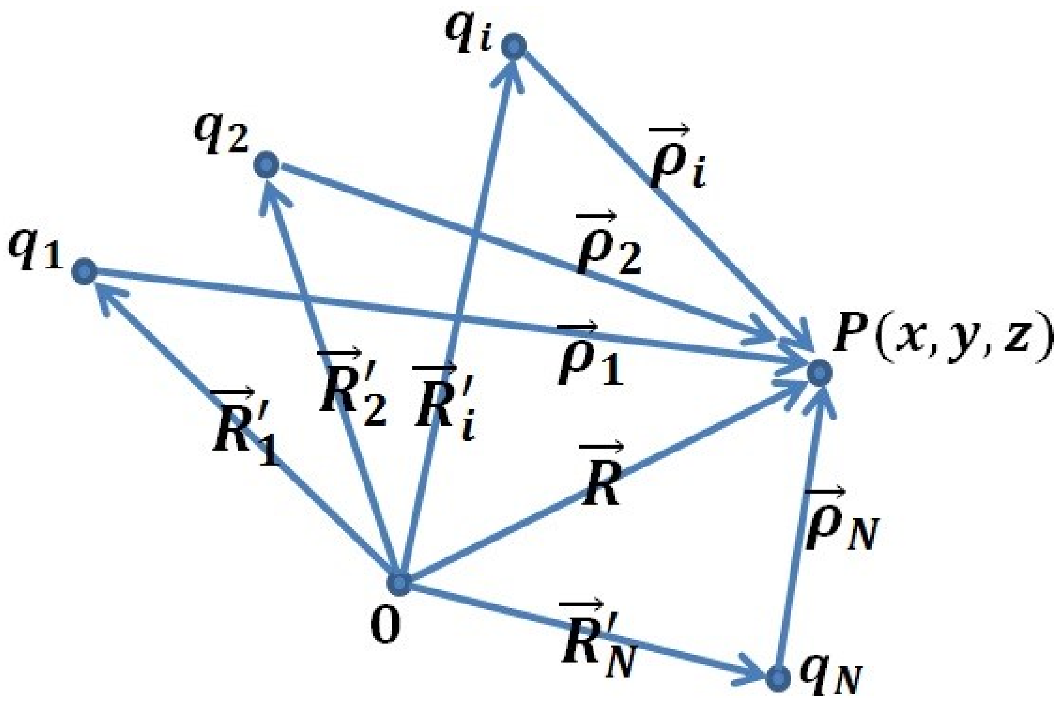
$$V(P) = \frac{1}{4\pi\epsilon_0}\int_{D'} \frac{dq'}{\left|\vec{R}-\vec{R}'\right|}$$

$$V(P) = \frac{1}{4\pi\epsilon_0}\int_{D'} \frac{\rho'_S dS'}{\left|\vec{R}-\vec{R}'\right|}$$



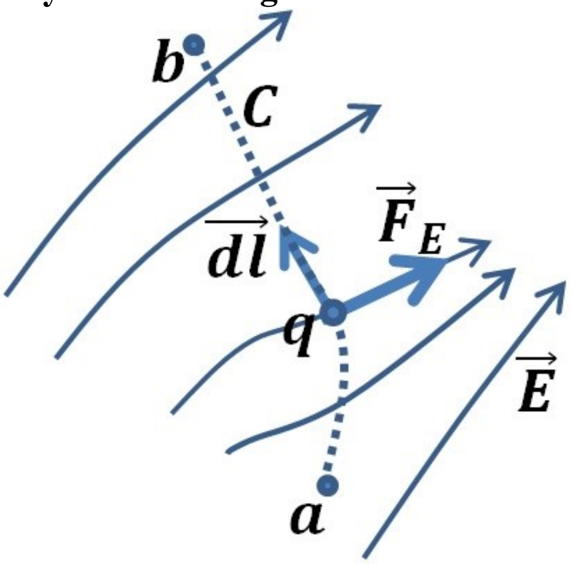
$$V(P) = \frac{1}{4\pi\epsilon_0}\int_{D'} \frac{dq'}{|\vec{R}-\vec{R}'|}$$

$$V(P) = \frac{1}{4\pi\epsilon_0}\int_{D'} \frac{\rho_l' dl'}{|\vec{R}-\vec{R}'|}$$



$$V(P) = \frac{1}{4\pi\epsilon_0}\sum_{i}^{N} \frac{q_i'}{|\vec{R}-\vec{R}'|}$$

Physical meaning of Electrostatic Potential



Charge q is moved from point 'a' to 'b' on contour C.

$\overrightarrow{dl}$ is tangent to C.

The force experienced by q is $\vec{F}_E$

$$\vec{F}_E = q\vec{E}$$

To maintain constant speed,

$$\vec{F}_{external} = -\vec{F}_E$$

Work done from 'a' to 'b'

$$W = \int_a^b \vec{F}_{external} \bullet \overrightarrow{dl}$$

$$W = \int_a^b -\vec{F}_E \bullet \overrightarrow{dl} = \int_a^b -q\vec{E} \bullet \overrightarrow{dl} \qquad = q\left(-\int_a^b \vec{E} \bullet \overrightarrow{dl}\right)$$

$$W = q\left(-\int_a^b -\nabla V \bullet \overrightarrow{dl}\right)$$

$$\nabla V \bullet \hat{l} = \frac{dV}{dl}$$

$$W = q\left(+\int_a^b \nabla V \bullet \hat{l} dl\right) = q\left(+\int_a^b \frac{dV}{dl} dl\right)$$

$$W = q\left(+\int_a^b dV\right) = q(V(b) - V(a))$$

$$W = q(\Delta V) = q\left(-\int_a^b \vec{E} \bullet \overrightarrow{dl}\right)$$

$$\Delta V = V(b) - V(a) = -\int_a^b \vec{E} \bullet \overrightarrow{dl}$$

ΔV is the work done to move a unit charge from 'a' to 'b' in the presence of $\vec{E}$

$\vec{E}$: Conservative field =>

$\vec{E}$: Conservative field => $\quad -\int_a^b \vec{E} \cdot \vec{dl}$ is path independent

$\vec{E}$: Conservative field => $\quad \oint_C \vec{E} \cdot \vec{dl} = 0$

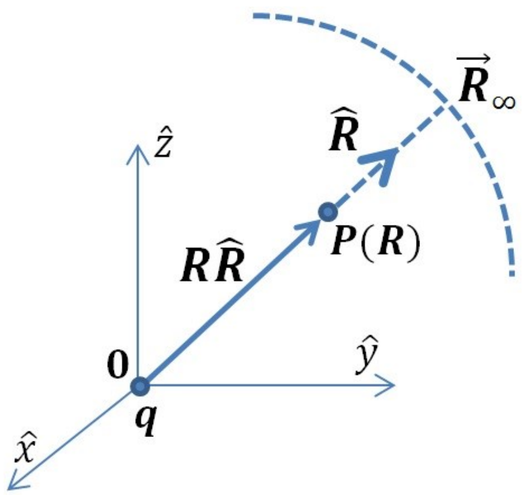
If a reference point, P_0 with a reference voltage, $V(P_0)$ is defined:

$$V(P) - V(P_0) = -\int_{P_0}^{P} \vec{E} \cdot \vec{dl}$$

$$V(P) - V_{ref} = -\int_{ref.}^{P} \vec{E} \cdot \vec{dl}$$

Absolute reference point: ∞ Absolute Potential: 0

$$V(P) - V(\infty) = V(P) - 0 = V(P) = -\int_{\infty}^{P} \vec{E} \cdot \vec{dl}$$



Localized charge at origin

It creates $V(P)$ at P

$$V(P) = -\int_{\infty}^{R} \vec{E} \cdot 1.dR.\hat{R}$$

$$V(P) = -\int_{\infty}^{R} \frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2} \cdot 1.dR.\hat{R}$$

$$V(P) = -\int_{\infty}^{R} \frac{q}{4\pi\epsilon_0}\frac{dR}{R^2} = \frac{q}{4\pi\epsilon_0}\frac{1}{R}\Bigg|_{\infty}^{R} \qquad \Bigg|_{\infty}^{R}$$

$$V(P) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{\infty}\right]$$

$$V(P) = \frac{q}{4\pi\epsilon_0 R}\ (Volts)$$