Lecture 6 :

Potentials and Fields : For a given time dependent charge ρ (**r**,t) and current **J**(**r**,t) Maxwell's equations are :

$$\text{(i)} \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \qquad \text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, \qquad \text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Since there are couplings between these differential equations obtaining their solutions for the general time dependent cases is not an easy job.

One can introduce the scalar V and the vector $\boldsymbol{A}$ potentials such that :

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}$$

Using these expressions for the fields one can easily obtain the following second order differential eqautions (still coupled ones)

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho;$$

$$\left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}.$$

Gauge transformations : One may change the scalar and vector potentials according to the transformations below and without changing the physical fields ***E*** and ***B***

$$\mathbf{A}' = \mathbf{A} + \nabla \lambda,$$

$$V' = V - \frac{\partial \lambda}{\partial t}.$$

here the function λ(**r**, t) is an arbitrary scalar function of position and time.

Conclusion : One may freely change his (her) potentials accordingly without changing the physically measurable fields in the lab. Then make use of this freedom such that one can get rid of the couplings between the scalar V and the vector $\boldsymbol{A}$ potentials

Coulomb gauge : $\nabla \cdot \boldsymbol{A} = 0$

This choice of gauge is mostly appropriate for the study of radiation problems and allow us to write down the Poisson's equation for the scalar potential. Its general solution is then :

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{\imath} d\tau'.$$

As to the vector potential, it can be found as a solution to the following equation :

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t} \right).$$

Lorentz gauge : $\nabla \cdot \boldsymbol{A} + \varepsilon_0 \mu_0 \frac{\partial V}{\partial t} = 0$

For a covariant treatment of the electrodynamics the Lorentz gauge choice is preferred. Then the uncoupled differential equations turn out to be inhomogenous wave equations :

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}.$$

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0} \rho.$$

We define the d'Alembertian operator $\square^2$:

$$\square^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \ :$$

So, using d'Alembertian operatör, our differential equations for the potentials become simpler :

$$\square^2 V = -\frac{1}{\varepsilon_0} \rho$$

$$\square^2 \boldsymbol{A} = -\mu_0 \mathbf{J}$$