



PROGRAMMING WITH MATLAB

WEEK 9





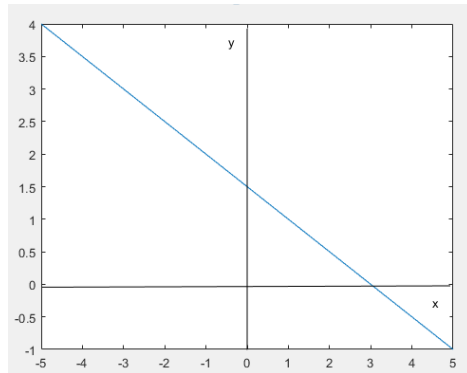
LINEAR ALGEBRAIC EQUATIONS



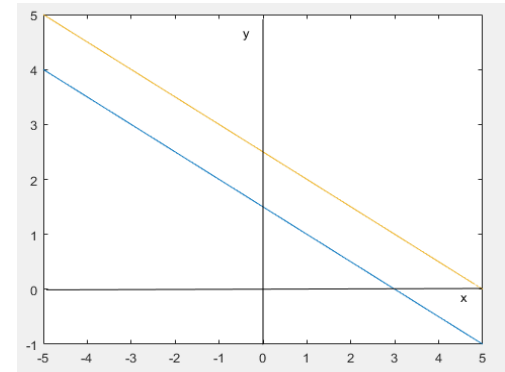
LINEAR ALGEBRAIC EQUATIONS

- Consider the two equations $ax+by=c$ and $dx+ey=f$. Since these equations represent two lines in the xy -plane, the simultaneous solution of these two equations (i.e. those points (x,y) that satisfy both equations) is merely the intersection of the two lines.
- Consider the following two sets of equations

$$\begin{aligned}x + 2y &= 3 \\ 3x + 6y &= 9\end{aligned}$$



$$\begin{aligned}\text{and } x + 2y &= 3 \\ x + 2y &= 5\end{aligned}$$



In the first set, the second equation is identical to the first equation, multiplied by 2. This case yields infinite solutions (singular system).

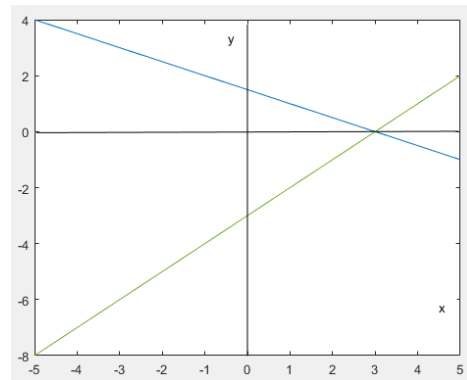
In the second set, the lines are parallel (but not identical), and this case yields no solution (singular system)

LINEAR ALGEBRAIC EQUATIONS

- But, the equations

$$x + 2y = 3$$

$$x - 2y = 3$$



This case yields one solutions (the lines are intersect at the $(3,0)$)

LINEAR ALGEBRAIC EQUATIONS

- It is often convenient to represent a system of equations as a matrix equation.
- Matrix notation enables us to represent multiple equations as a single matrix equation.
- Consider the system of equations:

$$x_1 + 3x_2 = 5$$

$$7x_1 - 11x_2 = 13$$

We may write the entire system as a matrix equation

$$\begin{bmatrix} 1 & 3 \\ 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$$

or as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 7 & -11 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$$

The 2x2 matrix \mathbf{A} is called the matrix of coefficients of the system of equations

LINEAR ALGEBRAIC EQUATIONS

- For square systems of equations (i.e. the number of equations and the unknowns are equal), the most powerful tool to determine the number of solutions the system has is determinant:
if $|A| = 0$, then there is no unique solution
- A $n \times n$ nonhomogeneous system of linear equations has a unique solution if and only if its determinant is non-zero.
- If its determinant is zero, then the system has either no solutions or an infinite number of solutions.

LINEAR ALGEBRAIC EQUATIONS

- The next step after the system of equations is written in matrix form is to solve $\mathbf{Ax} = \mathbf{b}$ equation for unknown x .
- Let's isolate x on one side of the equation

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

So we found the \mathbf{x} vector by multiplying the vector \mathbf{b} by the inverse of the matrix \mathbf{A}

The `inv (A)` command calculates the inverse of the matrix \mathbf{A}

LINEAR ALGEBRAIC EQUATIONS

- For example, consider the following three equations with three unknowns x_1 , x_2 , and x_3 :

$$x_1 - 3x_2 + 5x_3 = -7$$

$$-11x_1 + 13x_2 - 17x_3 = 19$$

$$23x_1 - 27x_2 + 31x_3 = -37$$

We write it in $\mathbf{Ax} = \mathbf{b}$ form

$$\begin{bmatrix} 1 & -3 & 5 \\ -11 & 13 & -17 \\ 23 & -27 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 19 \\ -37 \end{bmatrix}$$

There are two ways to solve this in MATLAB:

```
>> A = [1 -3 5;-11 13 -17;23 -27 31];
```

```
>> b = [7;19;37];
```

```
>> x = inv(A)*b
```

```
x =
```

```
1.3333
```

```
1.9048
```

```
-0.5238
```

or

```
>> x = A\b
```

```
x =
```

```
1.3333
```

```
1.9048
```

```
-0.5238
```


LINEAR ALGEBRAIC EQUATIONS

- We have seen that MATLAB is a built-in function (`inv`) to find the inverse of a matrix. There is also a built-in function (`det`) to find the determinant.

```
>> inv(A)
```

```
ans =
```

```
-0.6667 -0.5000 -0.1667  
-0.5952 -1.0000 -0.4524  
-0.0238 -0.5000 -0.2381
```

```
>> det(A)
```

```
ans =
```

```
84.0000
```

LINEAR ALGEBRAIC EQUATIONS

- The system of equations $\mathbf{Ax} = \mathbf{b}$ with m equations and n unknowns has solutions if and only if:

$$\text{rank}[\mathbf{A}] = \text{rank}[\mathbf{A} \ \mathbf{b}]$$

If the above condition is satisfied and $r = n$, then the solution is unique, where $r = \text{rank} [\mathbf{A}]$.

If the condition is satisfied but $r < n$, an infinite number of solutions exists.

If the system of equations is homogeneous ($\mathbf{Ax} = \mathbf{0}$), then $\text{rank} [\mathbf{A}] = \text{rank} [\mathbf{A} \ \mathbf{b}]$.

As a matter of fact, there is always the trivial solution $\mathbf{x} = \mathbf{0}$.

If $\text{rank}[\mathbf{A}] < n$, a nonzero solution exists.

If $m < n$, the homogeneous system of equations always has a nonzero solution.

If the number of equations is equal to the number of unknowns and $|\mathbf{A}| \neq 0$, then the system of equations has a solution and it is unique.

LINEAR ALGEBRAIC EQUATIONS

- **Practice:**
- Consider the following 2 X 2 system of equations:

$$x_1 + 2x_2 = 3$$

$$x_1 - 2x_2 = 3$$

Let's write these equations in the form $y = mx + b$.

$$y = -0.5x + 1.5$$

$$y = 0.5x - 1.5$$

We can plot these straight line equations in MATLAB:

```
>> x=-5:5;
```

```
>> y1 = -0.5*x + 1.5;
```

```
>> y2 = 0.5*x - 1.5;
```

```
>> plot(x, y1, x, y2)
```

LINEAR ALGEBRAIC EQUATIONS

The intersection of the lines is the point (3,0).

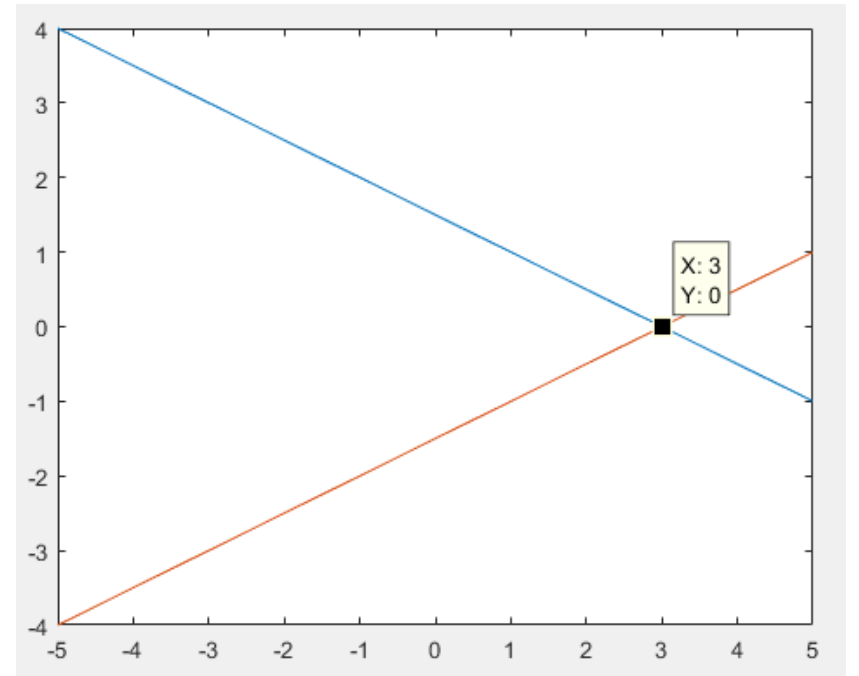
Now let's write this system of equations in matrix form:

$$\mathbf{A} \mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

We know the solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

So if we can find the inverse of the matrix \mathbf{A} , we can solve it.



LINEAR ALGEBRAIC EQUATIONS

First we compute the determinant of this matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 1 * -2 - 2 * 1 = -4$$

$$A^{-1} = \frac{1}{-4} \begin{bmatrix} -2 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$x_1 = 3$$

$$x_2 = 0$$