FEEDBACK CONTROL SYSTEMS

LECTURE NOTES-9/12

Frequency Response Technique

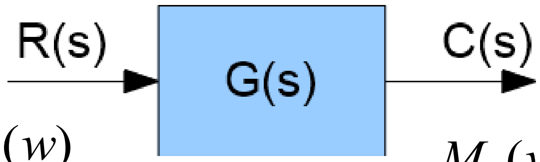
Frequency response methods, developed by Nyquist and Bode in the 1930s, are older than the root locus method, which was discovered by Evans in 1948 (Nyquist, 1932; Bode, 1945).

In the steady state, **sinusoidal inputs** to a linear system generate **sinusoidal responses** of the same frequency. Even though these responses are of the same frequency as the input, they **differ in amplitude and phase angle** from the input. **These differences are functions of frequency.**

$$M_o(w)\angle\phi_o(w) = M_i(w)M(w)\angle\left(\phi_i(w)+\phi(w)\right)$$

$$M(w) = \frac{M_o(w)}{M_i(w)}$$

$$M(w)\angle\phi(w)$$



$$M_i(w)\angle\phi_i(w)$$

$$M_o(w)\angle\phi_o(w)$$

$$\phi(w) = \phi_o(w) - \phi_i(w)$$

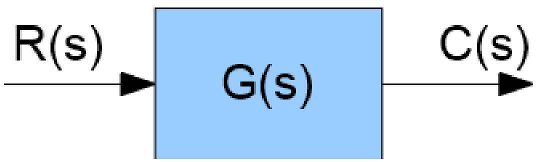
Sinusoids can be represented as complex numbers called ***phasors***

The **magnitude** of the complex number is the **amplitude** of the sinusoid, and the **angle** of the complex number is the **phase angle** of the sinusoid.

Thus, $M_1\cos(wt+\phi_1)$ can be represented as $M_1\angle\phi_1$ where the frequency, w is implicit

$$C(s) = \frac{As + Bw}{(s^2 + w^2)} G(s)$$

$$R(s) = \frac{As + Bw}{s^2 + w^2}$$



Represent as Phasor

$$M_i \angle \phi_i$$

$$M_i = \sqrt{A^2 + B^2}$$

$$\phi_i = -\tan^{-1}\left(\frac{B}{A}\right)$$

$$r(t) = A\cos wt + B\sin wt = \sqrt{A^2 + B^2}\cos(wt - \tan^{-1}(B/A))$$

$$C(s) = \frac{As + Bw}{(s + jw)(s - jw)} G(s)$$

$$C(s) = \frac{K1}{s + jw} + \frac{K2}{s - jw} + \text{partial fraction terms}$$

$$K_1 = \frac{As + Bw}{s - jw} G(s)\Big|_{s \to jw} = \frac{1}{2}(A + jB)G(-jw) = \frac{1}{2} M_i e^{-j\phi i} M_G e^{-j\phi_G}$$

$$K_1 = \frac{M_i M_G}{2} e^{-j(\phi_i + \phi_G)}$$

$$K_2 = \frac{As + Bw}{s + jw} G(s)\Big|_{s \to jw} = \frac{1}{2}(A - jB)G(-jw) = \frac{1}{2} M_i e^{j\phi i} M_G e^{j\phi_G}$$

$$K_2 = \frac{M_i M_G}{2} e^{j(\phi_i + \phi_G)} = K_1^*$$

$$M_G = |G(jw)|$$

$$\phi_G = \text{angle of G(jw)}$$

Steady State Output

$$C(s) = \frac{K1}{s + jw} + \frac{K2}{s - jw}$$

$$C(s) = \frac{\frac{M_i M_G}{2} e^{-j(\phi_i + \phi_G)}}{s + jw} + \frac{\frac{M_i M_G}{2} e^{j(\phi_i + \phi_G)}}{s - jw}$$

$$M_o \angle \phi_o = (M_1 \angle \phi_1)(M_G \angle \phi_G)$$

$$M_G \angle \phi_G = G(jw)$$

$$G(jw) = G(s)\Big|_{s \to jw}$$

$$c(t) = M_i M_G \left(\frac{e^{-j(wt + \phi_i + \phi_G)} + e^{j(wt + \phi_i + \phi_G)}}{2} \right)$$

$$c(t) = M_i M_G \cos(wt + \phi_i + \phi_G)$$

Asymptotic Approximations: Bode Plots

Simplified = approximated as a sequence of straight lines

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_k)}{s^m(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$|G(jw)| = \left.\frac{K|(s+z_1)||(s+z_2)|\dots|(s+z_k)|}{|s^m||(s+p_1)||(s+p_2)|\dots|(s+p_n)|}\right|_{s\to jw}$$

$$20\log|G(jw)| = 20\log K + 20\log|(s+z_1)| + 20\log|(s+z_2)|$$

$$\left.+\dots - 20\log|s^m| - 20\log|(s+p_1)| - \dots\right|_{s\to jw}$$

Magnitude of zero terms are added and pole terms are subtracted

If we knew the response of each term then the algebraic sum wield total response in dB

Graphical addition of straight lines is simplified

Bode Plots for G(s)=(s+a)

$$G(s) = s + a$$
$$s = jw$$
$$G(jw) = jw + a = a\left(j\frac{w}{a} + 1\right)$$

At low frequencies when w approaches to zero from w=0.01a to a

$$G(jw) \approx a$$
$$20\log M = 20\log a$$

At high frequencies where w>>a

$$G(jw) \approx a\left(\frac{jw}{a}\right) = a\left(\frac{w}{a}\right)\angle 90 = w\angle 90$$
$$20\log M = 20\log a + 20\log\frac{w}{a} = 20\log w$$

At break frequency (a) phase is 45
At low frequencies (0.1a) phase is 0
At high frequencies (10a) phase is 90

To make log-magnitude plot will be 0db in break frequency, it is convenient to Normalize the magnitude and scale the frequency

$$s + a$$

$$a\left[\left(\frac{s}{a}\right) + 1\right]$$

$$s_1 = \frac{s}{a}$$

$$s_1 + a$$

Quadratic Factor $$\frac{1}{1+2\zeta\left(j\frac{w}{wn}\right)+\left(j\frac{w}{wn}\right)^2}$$

$$G(jw)=\frac{1}{1+2\zeta\left(j\frac{w}{wn}\right)+\left(j\frac{w}{wn}\right)^2}$$

For low frequency such that w<<wn

$$20\log\left|\frac{1}{1+2\zeta\left(j\frac{w}{wn}\right)+\left(j\frac{w}{wn}\right)^2}\right|=-20\log\sqrt{\left(1-\frac{w^2}{wn^2}\right)^2+\left(2\zeta\frac{w}{wn}\right)^2}$$

$$w>>wn$$

$$-20\log\frac{w^2}{wn^2}=-40\log\frac{w}{wn}dB$$

$$-20\log 1=0dB$$

To form the composite sketch

- Arrange representation of transfer function so that DC gain of each element is unity (except for part that have poles or zeros at the origin) that absorb the gain into the overall gain

- Draw all component sketches

- Start at low frequency (DC) with the component that has the lowest frequency pole or zero

- Use this component to draw the sketch up to the frequency of the next pole/zero

- Change the slope of the sketch at this point to account for the new dynamics: -1 for pole +1 for zero for double poles

- Scale by DC gain

Example: Draw the bode diagram for the following transfer function:

$$G(jw) = \frac{10(jw+3)}{jw(jw+2)(jw^2 + jw + 2)}$$

$$G(jw) = \frac{7.5\left(\frac{jw}{3}+1\right)}{jw\left(\frac{jw}{2}+1\right)\left[\frac{jw^2}{2}+\frac{jw}{2}+1\right]}$$

$$7.5$$

$$\frac{1}{jw}$$

$$1 + j\frac{w}{3}$$

$$\left(1 + j\frac{w}{2}\right)^{-1}$$

$$\left[1 + j\frac{w}{2} + + \frac{jw^2}{2}\right]^{-1}$$

The corner frequencies of the third fourth and fifth terms are w=3, w=2 and w=$(2)^{0.5}$