ENE 206 – Fluid Mechanics

WEEK 5

- ## Introduction

➢ Fundamental physical principles are discussed:
i) conservation of mass
ii) conservation of momentum
iii) conservation of energy
iv) second law of thermodynamics

These laws are applied to system of fluid particles. The mathematical statements of these fundamental principles yield the integral form of governing flow equations, which are.
i) continuity equation
ii) momentum equation
iii) energy equation
iv) equation for the second law of thermodynamics.

- ## Conservation of mass

➢ **Continuity equation:** When the principle of conservation of mass is applied to a fluid flow, the resultant mathematical equation is the continuity equation. The application of the Reynolds' transport theorem with intensive property, $\eta = 1$ yields:

$$\frac{D}{dt}\int_{V_s}\rho dV = \frac{\partial}{\partial t}\int_{V}\rho\, dV + \int_{A}\rho\left(\vec{V}.\hat{n}\right)dA = 0 \qquad (4.1)$$

In Eqn. 4.1, the first term on the right hand side represents the rate of change of mass within the control volume, while the second term represents the rate of change of mass efflux through the control surface.

- ## Conservation of momentum

➢ **Conservation of linear momentum:** When the principle of conservation of momentum is applied to a fluid flow, the resultant mathematical equation is the linear momentum equation. The application of the Reynolds' transport theorem with intensive property, $\eta = \vec{V}$ yields:

$$\sum \vec{F} = \frac{D}{dt}\int_{V_s}\rho\vec{V}\, dV = \frac{\partial}{\partial t}\int_{V}\rho\vec{V}\, dV + \int_{A}\rho\vec{V}\left(\vec{V}.\hat{n}\right)dA \qquad (4.2)$$

In Eqn. 4.2, the first term on the right hand side represents the rate of change of linear momentum within the control volume, while the second term represents the rate of change of linear momentum efflux through the control surface. Here the total external forces acting on the control volume is the sum of the total force, $\sum \vec{F}_s$ and the total body force, $\sum \vec{F}_b$. Therefore the total force can be represented by a summation:

$$\sum \vec{F} = \sum \vec{F}_s + \sum \vec{F}_b \qquad (4.3)$$

Since eqn. 4.3 is in vector notation, its three scalar components in the Cartesian coordinate can be given as:

$$\sum F_x = \sum F_{sx} + \sum F_{bx} = \frac{\partial}{\partial t}\int_V \rho u dV + \int_A \rho u\left(\vec{V}.\hat{n}\right)dA \qquad (4.4a)$$

$$\sum F_y = \sum F_{sy} + \sum F_{by} = \frac{\partial}{\partial t}\int_V \rho v dV + \int_A \rho v\left(\vec{V}.\hat{n}\right)dA \qquad (4.4b)$$

$$\sum F_z = \sum F_{sz} + \sum F_{bz} = \frac{\partial}{\partial t}\int_V \rho w dV + \int_A \rho w\left(\vec{V}.\hat{n}\right)dA \qquad (4.4c)$$

For a steady flow, the partial derivative with respect to time are zero, that is $\partial/\partial t = 0$. Then eqn. 4.2 becomes:

$$\sum \vec{F} = \sum \vec{F}_s + \sum \vec{F}_b = \int_A \rho\vec{V}\left(\vec{V}.\hat{n}\right)dA \qquad (4.5)$$

For a steady and one-dimensional flow through a streamtube, eqn. 4.5 is further simplified.

- **Conservation of angular momentum:**

The equation for the conservation of angular momentum or the moment of linear momentum equation for an inertial frame is introduced. For this purpose, the system of particles is considered, and the cross product of Newton's second law of motion with the position vector, $\vec{r}$ is taken to yield:

$$\sum \vec{r}\ x\ \vec{F} = \vec{r}\ x\ \frac{D\vec{M}s}{dt} = \vec{r}\ x\ \frac{D\left(m_s\vec{V}\right)}{dt} \qquad (4.6)$$

This equation can be also written as follows:

$$\sum \vec{r}\ x\ \vec{F} = \frac{D\left[\vec{r}\ x\ m_s\vec{V}\right]}{dt} = \vec{r}\ x\ \frac{D\left[m_s\left(\vec{r}\ x\ \vec{V}\right)\right]}{dt} = \frac{D\vec{H}_s}{dt} \qquad (4.7)$$

where $\vec{H}_s$ is the angular momentum of the system of fluid particles such that

$$\vec{H}_s = \int_{m_s}\left(\vec{r}\ x\ \vec{V}\right)dm \qquad (4.8)$$

here $dm = \rho dV$, so that

Eqn. 4.7 can rearranged as:

$$\sum \vec{T} = \sum \vec{r}\ x\ \vec{F} = \frac{D}{dt}\int_{V_s} \rho\left(\vec{r}\ x\ \vec{V}\right)dV \qquad (4.9)$$

where $\sum \vec{T}$ is the sum of the moment of external forces acting on the system of fluid particles about the origin of the inertial reference frame. Finally, if Eqn. 4.9 is compared with the Reynolds' transport equation then it is possible to realize that the angular momentum is an extensive property and $\vec{r}\,x\vec{V}$ is an intensive property, that is $N = H_s$ and $\eta = \vec{r}\,x\vec{V}$. The application of Reynolds' transport theorem to Eqn. 4.9 with $\eta = \vec{r}\,x\vec{V}$ yields:

$$\sum \vec{T} = \frac{D}{dt}\int_{V_s} \rho\left(\vec{r}\,x\vec{V}\right)dV = \frac{\partial}{\partial t}\int_{V} \rho\left(\vec{r}\,x\vec{V}\right)\vec{V}dV + \int_{A} \rho\left(\vec{r}\,x\vec{V}\right)\left(\vec{V}.\hat{n}\right)dA \qquad (4.10)$$

Eqn. 4.10 is known as the equation for the conservation of angular momentum for an inertial control volume. The first term on the right hand side of the equation represents the rate of change of angular momentum within the control volume, while the second term represents the net rate of angular momentum efflux through the control surface.

Here the total moment of external forces acting on the control volume is the sum of the total moment of the surface forces $\sum \vec{T}_s$, and the total moment of body forces, $\sum \vec{T}_b$ about the origin of the same inertial frame. Therefore the total force can be represented by a summation:

$$\sum \vec{T} = \sum \vec{T}_s + \sum \vec{T}_b \qquad (4.11)$$

For a steady flow, the partial derivatives with respect to time are zero, that is that is $\partial/\partial t = 0$. Then eqn. 4.10 becomes:

$$\sum \vec{T} = \sum \vec{T}_s + \sum \vec{T}_b = \int_{A} \rho\left(\vec{r}\,x\vec{V}\right)\left(\vec{V}.\hat{n}\right)dA \qquad (4.12)$$

- **References**

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