2. NUMBERS

Example 1: Prove that $\sqrt{3}+\sqrt{5}$ is not a rational number.

Solution: Suppose that $\sqrt{3}+\sqrt{5}$ is a rational number. Then we can write

$$\sqrt{3}+\sqrt{5} \quad = \quad r \quad where \quad r \in \mathbb{Q} \ .$$

Now

$$\begin{aligned} \left(\sqrt{3}+\sqrt{5}\right)^2 \quad = \quad r^2 \quad \Rightarrow \quad 8+2\sqrt{15} \quad &= \quad r^2 \\ \Rightarrow \quad \underbrace{\sqrt{15}}_{irrasyonel} \quad &= \quad \underbrace{\frac{r^2-8}{2}}_{rational} \end{aligned}$$

which is a contraction. So, $\sqrt{3}+\sqrt{5}$ cannot be a rational number.
Example 2: Prova that there is no rational number whose cube is 2.
Solution: Assume that $\sqrt[3]{2}$ is a rational number. Then we can write

$$\sqrt[3]{2} \quad = \quad \tfrac{p}{q} \in \mathbb{Q} \quad , \quad (p,q) \quad = \quad 1 \quad .$$

Now

$$\tfrac{p^3}{q^3} \quad = 2 \quad \Longrightarrow \quad p^3 \quad = \quad \underbrace{2q^3}_{even} \ .$$

Since p^3 is an even number, p is an even number. Then, there exits a $k \in \mathbb{Z}$ such that $p = 2k$. On the other hand we have

$$\begin{aligned} p^3 \quad = \quad 2q^3 \quad \Longrightarrow \quad 8k^3 \quad &= 2q^3 \\ \Longrightarrow \quad 4k^3 \quad &= q^3 \end{aligned}$$

which means that q^3 is even, i.e. q is an even number. Thus we get $(p,q) = 2$ which is a contradiction.

Example 3: Prove that

$$3^n + 4^n \leq 5^n \text{ for all } n \in \mathbb{N}_2.$$

Solution:

The statement is true for $n = 2$ since $3^2+4^2 \quad = 25 \quad \leq \quad 5^2 \quad = 25.$ Assume that the statement true for $n = k$. i.e.

$$3^k + 4^k \leq 5^k.$$

If we multiply both hand sides by 5 then we have

$$\begin{array}{ccccccc} 3^k5 & + & 4^k5 & \leq & 5^{k+1} & & \\ 3^k3 & + & 4^k4 & \leq & 3^k3 & + & 4^k4 \quad \leq 5^k5 \\ 3^{k+1} & + & 4^{k+1} & < & 53^k & + & 54^k \quad \leq 5^{k+1}. \end{array}$$

Then the statement is true for $n = k+1$. Thus $3^n + 4^n \leq 5^n$ for all $n \in \mathbb{N}_2$ due to mathematical induction.