3. LINEAR POINT SETS

Definition: A subset of real numbers is said to be a linear point set.

Definition: Let A be a linear set and let u,v be a real number. If for any $x \in A$ we have $x \leq u$ then we say u is an upper bound of A and the set A is said to be upper bounded set. Similarly, if for any $x \in A$ we have $x \geq v$ then we say v is a lower bound of A and the set A is said to be lower bounded set. If a set is both bounded above and below it is called a bounded set.

Definition: *An element* of an upper bounded set A is said to be the maximum (of A) if it is greater than or equal to each element of the set. Maximum of A is denoted by $\max A$.

Definition: *An element* of a lower bounded set is said to be the minimum (of A) if it is less than or equal to each element of the set. Minimum of A is denoted by $\min A$.

Definition: The least upper bound of an upper bounded set A is said to be the supremum (of A). Supremum of A is denoted by supA.

Definition: The greatest lower bound of a lower bound set A is said to be the infimum (of A). Infimum of A is denoted by $\inf A$.

The following axiom guarentees the existence of supremum (infimum) of an upper bounded (a lower bounded) set:

Axiom of completeness: Every upper bounded subset of real numbers has a least upper bound.

Theorem: If the supremum of an upper bounded set belong to the set then it is the maximum and if the infimum of a lower bounded set belong to the set then it is the minimum.

Definition: Let A be a linear point set and let x is a real number. If in any deleted neighbourhood of x there exist a point of A then x said to be an accumulation point of A. Accumulation points of the set A is denoted by A'.

Example 1: Let

$$A = \left\{ \frac{(-1)^{n+1}}{n+1} : n \in \mathbb{N} \right\}.$$

Find $supA$,$infA$, $maksA$, $minA$, A'.

Solution: It is clear that $A = \left\{ \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, ... \right\}$. It is easy to see that

$$\inf A = \tfrac{1}{3} \in A \implies \min A = -\tfrac{1}{3}$$

$$\sup A = \tfrac{1}{2} \in A \implies maksA = \tfrac{1}{2}.$$

On the other hand for any n we can show that $\frac{(-1)^{n+1}}{n+1} \notin A^{'}$. Indeed; for

$$\begin{aligned} \varepsilon &= \left| \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+3}}{n+3} \right| \\ &= \left| \frac{1}{n+1} - \frac{1}{n+3} \right| \\ &= \left| \frac{2}{(n+1)(n+3)} \right| \\ &= \frac{2}{(n+1)(n+3)} \end{aligned}$$

we get

$$\left[\left(\frac{(-1)^{n+1}}{n+1} - \varepsilon - \frac{(-1)^{n+3}}{n+3} + \varepsilon \right) \cap A \right] \setminus \left\{ \frac{(-1)^{n+1}}{n+1} \right\} = \emptyset.$$

Moreover for any given $\varepsilon > 0$ there exists n such that $\frac{1}{n} < \varepsilon$. Hence $\frac{1}{n+1} < \varepsilon$. Since $\{(-\varepsilon, \varepsilon) \cap A\} \setminus \{0\} \neq \emptyset$, we get $0 \in A^{'}$. Thus $A^{'} = \{0\}$.

Example 2: Let S and T be nonempty bounded subsets of real numbers. Prove that

a) If $S \subseteq T$ then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

b) $\sup(S \cup T) = \max\{\sup S, \sup T\}$

Solution:

a) Since S and T are bounded and nonempty sets the supremum and infimum of them exist and $\inf S \leq \sup S$. So, it is enough to show that $\inf T \leq \inf S$ and $\sup S \leq \sup T$. For all $y \in T$ we know that $y \leq \sup T$. Since $\sup S$ is the least upper bound for S, we have $\sup T \geq \sup S$.

Similarly, to show that $\inf T \leq \inf S$, notice that $\inf T$ is a lower bound for T. Since $S \subseteq T$, we get $x \geq \inf T$ for all $x \in S$. Then $\inf T$ is a lower bound for S. Since $\inf S$ is the greatest lower bound of S, we get $\inf T \leq \inf S$.

b) From (a) one can have

$$\begin{aligned} \sup S &\leq \sup(S \cup T) \\ \sup T &\leq \sup(S \cup T). \end{aligned}$$

Thus, we get $maks\{\sup S, \sup T\} \leq \sup(S \cup T)$. Now for all $x \in S$ we can write $x \leq \sup S \leq maks\{\sup S, \sup T\}$ and for all $y \in T$ we can write $y \leq \sup T \leq maks\{\sup S, \sup T\}$. Let $a \in S \cup T$. As

$$\begin{aligned} a \in S \cup T &\implies a \in S \quad &or \quad a \in T \\ &\implies a \leq maks\{\sup S, \sup T\} \quad &or \quad a \leq maks\{\sup S, \sup T\} \end{aligned}$$

$maks\{\sup S, \sup T\}$ is an upper bound for $S \cup T$. From the definition; we get $\sup(S \cup T) \leq maks\{\sup S, \sup T\}$.

Greatest Integer Function

Definition: Let x be a real number. The greatest integer which is less than or equal to x is denoted by $[|x|]$.

Example 1: Solve the equation

$$[|3x|] = 3\,[|x|]$$

Solution: We can write

$$\begin{array}{rcl} x = [|x|] + t & , & 0 \leq t < 1 \\ 3x = 3\,[|x|] + 3t & , & 0 \leq 3t < 3 \end{array}$$

which implies

$$[|3x|] = \begin{cases} 3\,[|x|] & , \quad 0 \leq 3t < 1 \\ 3\,[|x|] + 1 & , \quad 1 \leq 3t < 2 \\ 3\,[|x|] + 2 & , \quad 2 \leq 3t < 3. \end{cases}$$

Then whenever $0 \leq 3t < 1$, $[|3x|] = 3\,[|x|]$ holds. This means that the fraction part of the number x is smaller than $\frac{1}{3}$, i.e., $[|3x|] = 3\,[|x|]$ if $0 \leq t < \frac{1}{3}$. Thus the solution set is $\underset{m\in\mathbb{Z}}{\cup} \left[m, m + \frac{1}{3}\right)$.