**4. FUNCTIONS-1**

**Definition:** Let $A$ and $B$ be two sets. Any subset of $A \times B$ is called a relation from $A$ to $B$.

Definition: A relation $f$ from $A$ to $B$ is said to be a function (well-defined) from $A$ to $B$ if for each $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$. In this case we write $f : A \to B$ and $f(a) = b$. The set $A$ is called the domain of $f$ and the set $B$ is the range of $f$.

Definition: Let $f : A \to B$ a function, let $E \subset A$ and let $H \subset B$. The set

$$f(E) := \{f(x) \in B : x \in E\} \subset B$$

is called the image of the set $A$. The set

$$f^{-1}(H) := \{x \in A : f(x) \in B\} \subset A$$

is called the pre-image of the set $B$.

**Example 1:** Let $f : X \to Y$ be a function. Show that

$$f(A \cup B) = f(A) \cup f(B)$$

for any $A, B \subset X$.

**Solution:** We are going to show that
**i)** $f(A \cup B) \subset f(A) \cup f(B)$
**ii)** $f(A) \cup f(B) \subset f(A \cup B)$.
$i)$ Let $y \in f(A \cup B)$. Then we have

$$\begin{aligned} \exists x &\in A \cup B \ni \ f(x) = y \\ &\Longrightarrow \exists x \in A \ni \ f(x) = y \text{ or } \exists x \in B \ni \ f(x) = y \\ &\Longrightarrow y \in f(A) \text{ or } y \in f(B). \end{aligned}$$

So, we get $f(A \cup B) \subset f(A) \cup f(B)$.

$ii)$ Since $A \subset A \cup B$, we get $f(A) \subset f(A \cup B)$. Similarly, as $B \subset A \cup B$, we get $f(B) \subset f(A \cup B)$. Then $f(A) \cup f(B) \subset f(A \cup B)$.

Finally, from $(i)$ and $(ii)$ the equality is obtained.

**Example 2:** Let $f : [0, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x} - 1$ and let $A = [0, 1) \cup \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$. Determine the following sets:

**a)** $f(A)$ **b)** $f^{-1}(A)$.

**Solution:**

$a)$ We know that $f(A \cup B) = f(A) \cup f(B)$. Then we have

$$\begin{aligned} f([0,1)) &= \{f(x) : x \in [0,1)\} \\ &= \{\sqrt{x} - 1 : 0 \le x < 1\} \\ &= \{\sqrt{x} - 1 : 0 \le \sqrt{x} < 1\} \\ &= \{\sqrt{x} - 1 : -1 \le \sqrt{x} - 1 < 0\} \\ &= [-1, 0). \end{aligned}$$

and

$$\begin{aligned} f\left(\left\{1+\tfrac{1}{n}:n\in\mathbb{N}\right\}\right) &= \left\{f(x):x\in\left\{1+\tfrac{1}{n}:n\in\mathbb{N}\right\}\right\} \\ &= \left\{\sqrt{x}-1:\exists n\in\mathbb{N}\ni x=1+\tfrac{1}{n}\right\} \\ &= \left\{\sqrt{1+\tfrac{1}{n}}-1:n\in\mathbb{N}\right\}. \end{aligned}$$

Hence $f(A)=[-1,0)\cup\left\{\sqrt{1+\frac{1}{n}}-1:n\in\mathbb{N}\right\}$.

*b)* We know that $f^{-1}(A)=f^{-1}(A)\cup f^{-1}(B)$. Then we can write

$$\begin{aligned} f^{-1}([0,1)) &= \{x\in[0,\infty):f(x)\in[0,1)\} \\ &= \{x\in[0,\infty):0\le\sqrt{x}-1<1\} \\ &= \{x\in[0,\infty):1\le\sqrt{x}<2\} \\ &= \{x\in[0,\infty):1\le x<4\} \\ &= [1,4). \end{aligned}$$

and

$$\begin{aligned} f^{-1}\left(\left\{1+\tfrac{1}{n}:n\in\mathbb{N}\right\}\right) &= \left\{x\in[0,\infty):f(x)\in\left\{1+\tfrac{1}{n}:n\in\mathbb{N}\right\}\right\} \\ &= \left\{x\in[0,\infty):\exists n\in\mathbb{N}\ni f(x)=1+\tfrac{1}{n}\right\} \\ &= \left\{x\in[0,\infty):\exists n\in\mathbb{N}\ni\sqrt{x}-1=1+\tfrac{1}{n}\right\} \\ &= \left\{x\in[0,\infty):\exists n\in\mathbb{N}\ni x=\left(2+\tfrac{1}{n}\right)^2\right\} \\ &= \left\{\left(2+\tfrac{1}{n}\right)^2:n\in\mathbb{N}\right\}. \end{aligned}$$

Hence $f^{-1}(A)=[1,4)\cup\left\{\left(2+\frac{1}{n}\right)^2:n\in\mathbb{N}\right\}$.

**Example 3:** Find the domain of the each of the following functions:

*a)* $f\left(x=\sqrt{x-1}+\sqrt{5-x}\right)$ *b)* $f(x)=\sqrt{1-\sqrt{9-x^2}}$

**Solution:**

*a)* $x-1\ge0$ and $5-x\ge0$ imply $D_f=[1,5]$.

*b)* As

$$\begin{aligned} 1-\sqrt{9-x^2}\ge0 &\Leftrightarrow \sqrt{9-x^2}\le1 \\ &\Leftrightarrow 0\le9-x^2\le1 \\ &\Leftrightarrow x^2\le9 \text{ and } x^2\ge8 \\ &\Leftrightarrow -3\le x\le3 \text{ and x}\ge2\sqrt{2} \text{ and x}\le-2\sqrt{2}. \end{aligned}$$

we have $D_f=\left[-3,-2\sqrt{2}\right]\cup\left[2\sqrt{2},3\right]$.