5. FUNCTIONS-2

Definition: Let $f : A \to B$ be a function.
i) If for any $x \neq y \in A$ we have $f(x) \neq f(y)$ then we say that f is injective (function) or injection.
ii) If for any $y \in B$ there exists $x \in A$ such that $f(x) = y$ then we say that f is surjective (function) or surjection.
iii) If f is both injective and surjective then it is said to be a bijective (function) or bijection.

Example 1: Let $f : [0,2] \to [-2,0]$ be a function defined by $f(x) = -\sqrt{4-x^2}$. Show that f is bijective.

Solution: If $f(x_1) = f(x_2)$ then we have

$$\begin{aligned} -\sqrt{4-x_1^2} = -\sqrt{4-x_2^2} \quad &\Rightarrow \quad x_1^2 = x_2^2 \\ &\Rightarrow \quad x_1 = x_2 \quad x \in [0,2]. \end{aligned}$$

So, f is an injective function.
It is not difficult to show that f is a surjection.

Example 2: Let $f : X \to Y$ and $g : Y \to X$ be two functions and let $fog = I_Y$. Show that
i) f is surjective.
ii) g is injective.

Solution: i) We show that for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$. Since g is well-defined there exists $x \in X$ such that $g(y) = x$ for all $y \in Y$. Now, since f is well-defined, for all $x \in X$ there exists $z \in Y$ such that $f(x) = z$. Then one can have

$$\begin{aligned} & f(g(y)) = fog(y) = z \\ & \Rightarrow I_Y(y) = z \\ & \Rightarrow y = z \\ & \Rightarrow y = f(x). \end{aligned}$$

So, for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
ii) Let $g(y_1) = g(y_2)$. Since $g(y_1)$ and $g(y_2) \in X$ and f is well-defined we can write

$$f(g(y_1)) = fg(y_2) \Rightarrow I_Y(y_1) = I_Y(y_2) \Rightarrow y_1 = y_2.$$