7. FUNCTIONS-4

Example 1: Investigate whether the following functions are periodic. Find the essential periods.

$$a)\ f(x) = \sin x^2 \quad b)\ f(x) = \cos^2 x$$

Solution:

a) For any x we have

$$\begin{aligned}
&f(x+p) = \sin(x+p)^2 = \sin x^2 = f(x) \\
&\Rightarrow \quad (x+p)^2 = x^2 + 2k\pi \quad or \quad (x+p)^2 = \pi - x^2 + 2k\pi;\ k \in \mathbb{Z} \\
&\Rightarrow \quad 2xp + p^2 = 2k\pi \quad or \quad 2x^2 + 2xp + p^2 = 2k\pi
\end{aligned}$$

Hence f is not periodic. Because, period can not depend on x.

b) For any x we have

$$\begin{aligned}
&f(x+p) = \cos(x+p)^2 = \cos x^2 = f(x) \\
&\Rightarrow \cos(x+p) = \pm\cos x \\
&\Rightarrow \cos(x+p) = \cos(x) \text{ or } \cos(x+p) = -\cos x \\
&\Rightarrow \cos x \cos p - \sin x \sin p = \cos x \text{ or } \cos x \cos p - \sin x \sin p = -\cos x \\
&\Rightarrow p = 2k\pi \text{ or } p = (2k-1)\pi;\ \ k \in \mathbb{Z}.
\end{aligned}$$

Hence f is periodic and the esseantial period of f is π.

Example 2: Find the domain of each of the following functions.

$$a)\ f(x) = \sqrt{9-x^2} + \log\left(\tfrac{x+1}{x-2}\right) \quad b)\ f(x) = \arccos x - \arcsin(3-x).$$

Solution:

a) Since

$$\mathcal{D}\left(\sqrt{9-x^2}\right) = \left\{x : 9 - x^2 \geq 0\right\} = [-3,3]$$

and

$$\mathcal{D}\left(\log\left(\frac{x+1}{x-2}\right)\right) = \left\{x : \frac{x+1}{x-2} > 0\right\} = (-\infty,-1) \cup (2,\infty)$$

we get $\mathcal{D}(f) = \mathcal{D}\left(\sqrt{9-x^2}\right) \cap \mathcal{D}\left(\log\left(\frac{x+1}{x-2}\right)\right) = [-3,-1) \cup (2,3]$.

b) Since

$$\mathcal{D}(arccosx) = \{x : -1 \leq x \leq 1\} = [-1,1]$$

and

$$\mathcal{D}(\arcsin(3-x)) = \{x : -1 \leq 3 - x \leq 1\} = [2,4]$$

we get $\mathcal{D}(f) = \mathcal{D}(arccosx) \cap \mathcal{D}(\arcsin(3-x)) = \emptyset$.

Example 3: Show that the following equalities hold.

a) $arccos(-x) = \pi - \arccos x$; $x \in [-1,1]$

b) $arcsinx = arccos\sqrt{1-x^2}$; $x \in [0,1]$.

Solution:
a) Let $x \in [-1,1]$. If $arccos\,(-x) = \alpha$ then we have

$$\begin{aligned} cos\alpha = -x \quad &\Rightarrow \quad cos\,(\pi - \alpha) = x \quad , \quad 0 \le \pi - \alpha \le \pi. \\ &\Rightarrow \quad \pi - \alpha = \arccos x \\ &\Rightarrow \quad \alpha = \pi - \arccos x \\ &\Rightarrow \quad arccos\,(-x) = \pi - \arccos x \quad . \end{aligned}$$

b) Let $0 \le x \le 1$ and $arcsinx = \alpha$. Then one cane have $x = sin\alpha$, $0 \le \alpha \le \frac{\pi}{2}$. We finally have

$$\begin{aligned} \cos\alpha \quad = \quad \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2} \quad &\Rightarrow \quad \alpha = \arccos\sqrt{1-x^2} \\ &\Rightarrow \quad \arcsin x = \arccos\sqrt{1-x^2}. \end{aligned}$$