7. SEQUENCES AND LIMIT

Definition: A function whose domain is $\mathbb{N}$ is called a sequence.

In this note we study with real termed sequences (the range will be real numbers).

Definition: Let $x = (x_n)$ be a sequence and L be a real number. If for any $\varepsilon > 0$ there exists n_0 such that $|x_n - L| < \varepsilon$ whenever $n \geq n_0$ then we say that (x_n) is convergent to L. In this case (x_n) is called convergent and L is called the limit of (x_n) and we write $\lim_{n\to\infty} x_n = L$ or $x_n \to L$.

Remark: If a sequence is not convergent then it is said to be divergent.

Theorem: A monotone sequence is convergent if and only if it is bounded.

Example 1: Find the limit of each of the following sequences:

$$a)\ (a_n) = \left(\frac{n^2-2n+3}{5n^3}\right) \quad b)\ (b_n) = \left(1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + ... + (-1)^{n-1}\frac{1}{3^{n-1}}\right)$$

Solution:

a) We have $\lim_{n\to\infty} \frac{n^2-2n+3}{5n^3} = \lim_{n\to\infty} \frac{1}{5n} - \lim_{n\to\infty} \frac{2}{5n^2} + \lim_{n\to\infty} \frac{3}{5n^3} = 0$.

b) We get

$$\begin{aligned}
\lim_{n\to\infty}\left(1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + ... + (-1)^{n-1}\frac{1}{3^{n-1}}\right) &= \lim_{n\to\infty} \frac{1-\left(-\frac{1}{3}\right)^n}{1-\left(-\frac{1}{3}\right)} \\
&= \lim_{n\to\infty} \frac{3\left[1-\left(-\frac{1}{3}\right)^n\right]}{4} \\
&= \frac{3}{4}
\end{aligned}$$

Example 2: Let (x_n) be the sequence defined by $x_1 = 1$, $x_n = \frac{2}{5-x_{n-1}}$ for $n \geq 2$. Find $\lim_n x_n$.

Solution: We can use the theorem above. Firstly, we will show that (x_n) is monotone. Since $x_1 = 1$ and $x_2 = \frac{1}{2}$ we have $x_1 > x_2$. Now let $x_k > x_{k+1}$. Then we have

$$\begin{aligned}
x_k > x_{k+1} &\Rightarrow -x_k < -x_{k+1} \\
&\Rightarrow 5 - x_k < 5 - x_{k+1} \\
&\Rightarrow \frac{2}{5-x_k} > \frac{2}{5-x_{k+1}} \\
&\Rightarrow x_{k+1} > x_{k+2}.
\end{aligned}$$

Hence, from mathematical induction we get that (x_n) is a monoton decreasing sequence. Since (x_n) sequence is monotonic decreasing it is bounded above and $x_1 = 1$ is an upper bound. We show that $x_n > 0$ for all $n \geq 2$. For this purpose let $x_k > 0$. Then we get

$$
\begin{aligned}
x_k > 0 \quad &\Rightarrow \quad -x_k < 0 \\
&\Rightarrow \quad 5 - x_k < 5 \\
&\Rightarrow \quad \tfrac{2}{5} < \tfrac{2}{5-x_k} \\
&\Rightarrow \quad 0 < x_{k+1} \qquad .
\end{aligned}
$$

Then (x_n) is bounded below. Since it is bounde both above and below, it is bounded. Thus the limit exits.
Let $x_n \to L$. Then $x_{n-1} \to L$. So we have

$$
\begin{aligned}
x_n = \tfrac{2}{5-x_{n-1}} \quad &\Rightarrow \quad \lim_n x_n = \lim_n \tfrac{2}{5-x_{n-1}} \\
&\Rightarrow \quad L = \tfrac{2}{5-L} \\
&\Rightarrow \quad L_{1,2} = \tfrac{5 \pm \sqrt{17}}{2}.
\end{aligned}
$$

Since $0 < x_n \leq 1$ for all n, $0 < L \leq 1$ we have $L = \frac{5-\sqrt{17}}{2}$.

Example 3: Calculate the following limits:

a) $\lim_n (2^n + 3^n)^{\frac{1}{n}}$

b) $\lim_n \frac{1}{\sqrt{n}} \left(1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}\right)$.

Solution:

a) For all $n \in \mathbb{N}$ we can write

$$
\begin{aligned}
3^n \leq 2^n + 3^n \leq 3^n + 3^n \quad &\Rightarrow \quad 3^n \leq 2^n + 3^n \leq 2 3^n \\
&\Rightarrow \quad 3 \leq (2^n + 3^n)^{\frac{1}{n}} \leq 2^{\frac{1}{n}} 3.
\end{aligned}
$$

Since $\lim_n 3 = 3$ and $\lim_n 2^{\frac{1}{n}}.3 = 3$, from the Sandvich Theorem

$$
\lim_n (2^n + 3^n)^{\frac{1}{n}} = 3.
$$

b) Let $x_n = 1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$ and $y_n = \sqrt{n}$. Note that (y_n) is increasing and $y_n \to \infty$. From Stolz Theorem we obtain

$$
\begin{aligned}
\lim_n \tfrac{x_n}{y_n} \quad &= \lim_n \tfrac{x_n - x_{n-1}}{y_n - y_{n-1}} \\
&= \lim_n \tfrac{\left(1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}\right) - \left(1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n-1}}\right)}{\sqrt{n} - \sqrt{n-1}} \\
&= \lim_n \tfrac{\frac{1}{\sqrt{n}}}{\sqrt{n} - \sqrt{n-1}} \\
&= \lim_n \tfrac{\sqrt{n} + \sqrt{n+1}}{\sqrt{n}(n-(n-1))} \\
&= \lim_n \left(1 + \sqrt{\tfrac{n+1}{n}}\right) = 2.
\end{aligned}
$$