8. LIMIT OF FUNCTIONS

Definition: Let $f : A \to \mathbb{R}$ be a function and let $x_0 \in A'$. If for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - x_0| < \delta$ we say that the limit of f at point x_0 is L. In this case we write $\lim\limits_{x \to x_0} f(x) = L$.

Definition: Let A be an unbounded subset of real numbers and let $f : A \to \mathbb{R}$ be a function. If for any $\varepsilon > 0$ there exists $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \geq M$ then we say that the limit of f as x approaching infinity is equal to L. In yhis case we write $\lim\limits_{x \to \infty} f(x) = L$. Similar definition can be given for $\lim\limits_{x \to -\infty} f(x) = L$.

Definition: Let $f : A \to \mathbb{R}$ be a function and let $x_0 \in A'$. If for any $M > 0$ there exists $\delta > 0$ such that $f(x) > M$ whenever $|x - x_0| < \delta$ then we say that f diverges infinity as x approaching x_0. In this case we write $\lim\limits_{x \to x_0} f(x) = \infty$. Similar definition can be given for $\lim\limits_{x \to x_0} f(x) = -\infty$.

Remark: If we use $x_o < x < x_0 + \delta$ or $x_0 - \delta < x < x_0$ instead of $|x - x_0| < \delta$ in the definitions above we define the sided limits at x_0. These concepts are called right handed and left handed limit, respectively. In these cases we write $\lim\limits_{x \to x_0^+} f(x) = L$ and $\lim\limits_{x \to x_0^-} f(x) = L$. Of course L can be infinity.

Theoem: $\lim\limits_{x \to x_0} f(x) = L$ iff $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x) = L$.

Example 1: Usıng the definition of limit prove that

$$\lim_{x \to 0} (2x + 1) = 1.$$

Solution: We show that for all $\varepsilon > 0$ there exits a $\delta > 0$ such that $|(2x + 1) - 1| < \varepsilon$ whenever $|x| < \delta$. Let $\varepsilon > 0$. If we choose $\delta = \varepsilon/2$ we get

$$\begin{aligned} |(2x+1) - 1| &= 2\,|x| \\ &< 2\delta = \varepsilon \end{aligned}$$

whenever $|x| < \delta$.

Example 2: Evaluate the following limits.

$$a) \quad \lim_{x \to 0} \frac{|x|}{x} \qquad b) \quad \lim_{x \to 2} \frac{(-1)^{[|x|]+1}}{x - 2}$$

Solution

a) As $\lim\limits_{x \to 0^+} \dfrac{|x|}{x} = 1$ and $\lim\limits_{x \to 0^-} \dfrac{|x|}{x} = -1$ the limit does not exist

b) Since $\lim\limits_{x \to 2^-} \frac{(-1)^{[|x|]+1}}{x-2} = -\infty$ and $\lim\limits_{x \to 2^+} \frac{(-1)^{[|x|]+1}}{x-2} = -\infty$ we have $\lim\limits_{x \to 2} \frac{(-1)^{[|x|]+1}}{x-2} = -\infty$.

Example 3: Calculate the following limits

a) $\lim\limits_{x\to 0}$

b) $\lim\limits_{n}\frac{2x^2-3x-4}{\sqrt{x^4+1}}$.

Solution:

a)

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan x-\sin x}{x^3} &= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}\\
&= \lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\frac{\left(2\sin^2\frac{x}{2}\right)}{x^2\cos x}\\
&= \frac{1}{2}.
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{x\to\infty}\frac{2x^2-3x-4}{\sqrt{x^4+1}} &= \lim_{x\to\infty}\frac{x^2\left(2-\frac{3}{x}-\frac{4}{x^2}\right)}{x^2\sqrt{\frac{1}{x^4}+1}}\\
&= \frac{1}{2}.
\end{aligned}$$