9. CONTINUITY

Definition: Let $f : A \to \mathbb{R}$ be a function and let $x_0 \in A$. If for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$ we say that f is continuous at point x_0.

Remark: If we use $x_o < x < x_0+\delta$ or $x_0-\delta < x < x_0$ instead of $|x - x_0| < \delta$ in the definition above we define the continuity from right and left, respectively.

Theorem: f is continuous at x_0 if and only if it is continuous both from right and left.

If we consider the definition of the concept of the limit we can get the following corollary:

Corollary: f is continuous at x_0 if and only if $\lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x) = f(x_0)$.

Remark: It is clear from the definition that a function f is continuous at a point that is in the domain and is not an accumulation point. Therefore, we should investigate the continuity of a function at the points of the domain that are accumulation points.

Definition: A point where f is not continuous is called a discontinuity point.

Definition: Let $f : A \to \mathbb{R}$ be a function. If for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(t)| < \varepsilon$ whenever $|x - t| < \delta$ we say that f is uniformly continuous over A.

Note that if f is uniformly continuous then it is continuous over the domain.

Example 1: Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 5$ is continuous at $x_0 = 2$.

Solution: We will show that for all $\varepsilon > 0$ there exits a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

For all $\varepsilon > 0$ if we choose $\delta = \frac{\varepsilon}{2}$ we have

$$\begin{aligned} |f(x) - f(2)| &= |2x + 5 - 9| \\ &= 2|x - 2| \\ &< 2\delta = \varepsilon. \end{aligned}$$

Example 2: If the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin x & , \quad x \leq \frac{\pi}{2} \\ ax + \pi & , \quad x > \frac{\pi}{2} \end{cases}$$

is continuous everywhere then determine a.

Solution: It is obvious that f is continuous for $x \neq \dfrac{\pi}{2}$. If f is continuous at $x_0 = \dfrac{\pi}{2}$ then we know that $\lim\limits_{x\to\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to\frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$. Since

$$\begin{aligned} \lim_{x\to\frac{\pi}{2}^-} f(x) &= \lim_{x\to\frac{\pi}{2}^-} \sin x = 1 \\ \lim_{x\to\frac{\pi}{2}^+} f(x) &= \lim_{x\to\frac{\pi}{2}^+} f(x) = a\frac{\pi}{2} + \pi \end{aligned}$$

we get $a=\frac{2-2\pi}{\pi}$.
Example 3: Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(x)=x^2+sgn\left(x^2-1\right).$$

Determine the points of discontinuity. For each point of dicontinuity determine types of discontinuities.

Solution: We can write

$$\begin{aligned} f(x) &= \begin{cases} x^2-1 &, \quad x^2<1 \\ x^2 &, \quad x^2=1 \\ x^2+1 &, \quad x^2>1 \end{cases} \\ &= \begin{cases} x^2+1 &, \quad -\infty<x<-1 \\ x^2 &, \quad x=-1 \\ x^2-1 &, \quad -1<x<1 \\ x^2 &, \quad x=1 \\ x^2+1 &, \quad x>1. \end{cases} \end{aligned}$$

It is clear that f is continuous except ∓ 1. As

$$\begin{aligned} \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} x^2+1=2 \\ \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} x^2-1=0. \end{aligned}$$

f is discontinuous at $x=-1$. The type of discontinuity is jump discontinuity. Since

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} x^2-1=0 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} x^2+1=2. \end{aligned}$$

f is discontinuous at $x=1$. The type of discontinuity is again jump discontinuity.

Example 4: Is the function f defined by $f(x)=x\sin x$ is uniformly continuous on $\mathbb{R}$?

Solution: Let $(x_n)=(n\pi)$ and $(y_n)=\left(n\pi+\frac{1}{n}\right)$. We get

$$|x_n-y_n|=\frac{1}{n}\to 0,(n\to\infty).$$

Then for any $\varepsilon>0$ there exists n_0 such that $|x_n-y_n|<\delta$ whenever $n\geq n_0$. On the other hand for all $n\geq n_0$

$$
\begin{aligned}
|f(x_n) - f(y_n)| &= |x_n \sin x_n - y_n \sin y_n| \\
&= \left| n\pi \sin n\pi - \left(n\pi + \tfrac{1}{n}\right) \sin\left(n\pi + \tfrac{1}{n}\right) \right| \\
&= \left| \left(n\pi + \tfrac{1}{n}\right) \left(\sin n\pi \cos \tfrac{1}{n} + \sin \tfrac{1}{n} \cos n\pi\right) \right| \\
&= \left(n\pi + \tfrac{1}{n}\right) \sin \tfrac{1}{n} \\
&= \pi \frac{\sin \frac{1}{n}}{\frac{1}{n}} + \tfrac{1}{n} \sin \tfrac{1}{n} \\
&\to \pi, \quad (n \to \infty).
\end{aligned}
$$

Then for $\varepsilon = 1$ we can write $|f(x_n) - f(y_n)| > 1$. Hence, f is not uniformly countinuous.