11. DERIVATIVE

Definition: Let $f : A \to \mathbb{R}$ be a function and let $x_0 \in A \cap A'$. If the limit

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists then f is said to be differentiable at x_0. The limit in this definition is equivalent to the following one:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Theorem: Let $f : A \to \mathbb{R}$ be a function and let $x_0 \in A \cap A'$. If f is differentiable at x_0 then f is continuous at x_0.

Example 1: Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 2x^3 & , \quad x \geq 0 \\ 3x^2 & , \quad x < 0 \end{cases}$. Is f differentiable at $x_0 = 0$?

Solution: We have

$$\begin{aligned} f'(0^+) &= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{2x^3 - 0}{x - 0} = 0 \\ f'(0^-) &= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{3x^2 - 0}{x - 0} = 0. \end{aligned}$$

Hence $f'(0) = 0$.

Example 2: Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \begin{cases} \sin \frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$. Is f differentiable at $x = 0$?

Solution: Since $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist f is not continuous at $x = 0$. Therefore, f is not differentiable at $x = 0$.

Example 3: Calculate the derivative of the following functions:

$$\begin{aligned} &a) \quad f(x) = \sqrt{\sin^5 x^3} + \tan^2 4x \\ &b) \quad f(x) = 2 \arctan \sqrt{\tfrac{2x+1}{3}} + \ln \left(\tfrac{1}{\sqrt{x^3+2}} \right) \end{aligned}$$

Solution:

a)

$$f(x) = \sqrt{\sin^5 x^3} + \tan^2 4x \quad \Rightarrow \quad \begin{aligned} f'(x) &= \tfrac{5}{2} \left(\sin x^3 \right)^{3/2} \cos x^3 \left(3x^2 \right) + 8 \tan 4x \left(1 + \tan^2 4x \right) \\ &= \tfrac{15}{2} x^2 \left(\sin x^3 \right)^{3/2} \cos x^3 + 8 \tan 4x \left(1 + \tan^2 4x \right) \end{aligned}$$

b)

$$f(x) = 2 \arctan \sqrt{\tfrac{2x+1}{3}} + \ln \left(\tfrac{1}{\sqrt{x^3+2}} \right) \quad \Rightarrow \quad \begin{aligned} f'(x) &= 2 \tfrac{1}{1 + \frac{2x+1}{3}} \left(\tfrac{2x+1}{3} \right)^{-1/2} \tfrac{2}{3} + \tfrac{\frac{-1}{2} \left(x^3+2 \right)^{-3/2} 3x^2}{\left(x^3+2 \right)^{-1/2}} \\ &= \tfrac{1}{2+x} \left(\tfrac{2x+1}{3} \right)^{-1/2} + \tfrac{3x^2}{2(x^2+2)} \end{aligned}$$

Example 4: Calculate the derivative of the following functions:

$$a)\quad x^y = y^x$$
$$b)\quad \begin{cases} x = t^3 + 3t \\ y = t\arctan t - \ln\sqrt{1+t^2} \end{cases}, \quad t \in \mathbb{R}$$

Solution: a)

$$\begin{aligned} x^y = y^x \quad &\Rightarrow \quad y\ln x = x\ln y \\ &\Rightarrow \quad y^{'}\ln x + \tfrac{y}{x} = \ln y + \tfrac{xy^{'}}{y} \\ &\Rightarrow \quad y^{'} = \tfrac{\ln y - y/x}{\ln x - x/y} \end{aligned}$$

b)

$\dot{x} = 3t^2 + 3$

$\dot{y} = \arctan t + t\frac{1}{1+t^2} - \frac{\frac{1}{2}\frac{2t}{\sqrt{1+t^2}}}{\sqrt{1+t^2}} = \arctan t$

$y^{'} = \frac{\dot{y}}{\dot{x}} = \frac{\arctan t}{3t^2+3}$.

Example 5: Find the n^{th} derivative of the function f defined by $f(x) = \sqrt{x}$, $x > 0$.

Solution:

$$\begin{aligned} f(x) = (x)^{1/2} \quad &\Rightarrow \quad f^{'}(x) = \tfrac{1}{2}x^{\frac{-1}{2}} \\ &\Rightarrow \quad f^{''}(x) = \tfrac{1}{2}\tfrac{-1}{2}x^{\frac{-3}{2}} \\ &\Rightarrow \quad f^{'''}(x) = \tfrac{1}{2}\tfrac{-1}{2}\tfrac{-3}{2}x^{\frac{-5}{2}} \\ &\Rightarrow \quad f^{(iv)}(x) = \tfrac{1}{2}\tfrac{-1}{2}\tfrac{-3}{2}\tfrac{-5}{2}x^{\frac{-7}{2}} \\ &\qquad\quad . \\ &\Rightarrow \qquad . \\ &\qquad\quad . \\ &\Rightarrow \quad f^{(n)}(x) = \tfrac{(-1)^{n-1}1.3.5...(2n-3)}{2^n}x^{\left(\frac{2n-1}{2}\right)} \end{aligned}$$