12. APPLICATIONS OF DERIVATIVE

Example 1: Consider the gunction defined by $f(x) = \dfrac{x}{x^2+1}$.

a) Determine the critical points.
b) Identify, the intervals on which f is increasing and decreasing.
c) Find the local maximum and local minimum points and values.

Solution:
$f(x) = \frac{x}{x^2+1} \quad \Rightarrow \quad f^{'}(x) = \frac{1-x^2}{(x^2+1)^2}$

a) $f^{'}(x) = 0 \Leftrightarrow x = \pm 1$. (critical points)
b) Since $f^{'}(x) > 0$ for $x \in (-1,1)$, f is increasing in $(-1,1)$.
Since $f^{'}(x) < 0$ for $x \in (-\infty, 1) \cup (1,\infty)$, f is decreasing in $(-\infty, 1) \cup (1,\infty)$.
c) $f^{''}(x) = \frac{2x^5-6x-4x^3}{(x^2+1)^4}$

$f^{''}(-1) = \frac{1}{2} > 0 \quad \Rightarrow \quad x = -1$ relative min point
$f^{''}(1) = \frac{-1}{2} < 0 \quad \Rightarrow \quad x = 1$ relative max point
$f(-1) = \frac{-1}{2}$ relative min value
$f(1) = \frac{1}{2}$ relative max value

Example 2: Find the equation of the tangent line to the curve $y = x^3 - 6x + 2$ that is parallel to the line $y = 6x - 2$.

Solution: If the tangents have to be parallel to the line then they must have the same gradient. i.e. slope should be 6.

$y = x^3 - 6x + 2 \Rightarrow y^{'} = 3x^2 - 6 = 6 \Rightarrow x = \pm 2$.
Now as $x = 2 \Rightarrow y = -2$, the equation $y + 2 = 6(x-2) \Rightarrow y = 6x - 14$ obtained and as $x = -2 \Rightarrow y = 6$ the equation $y - 6 = 6(x-2) \Rightarrow y = 6x = 18$ obtained.

Example 3: Show that the ineuality $n(b-a)a^{n-1} < b^n - a^n < n(b-a)b^{n-1}$ holds for any $0 < a < b$.

Solution: Define

$$\begin{aligned} f : [a,b] &\rightarrow \mathbb{R} \\ x &\rightarrow f(x) = x^n. \end{aligned}$$

f is differantiable in (a,b) and continuous in $[a,b]$. From the mean value theorem there exists $c \in (a,b)$ such that $f^{'}(x) = \frac{f(b)-f(a)}{b-a}$. So:

$f^{'}(x) = nx^{n-1}$; $f^{'}(c) = nc^{n-1} = \frac{b^n - a^n}{b-a}$. (1)

$$\begin{array}{lll} c \in (a,b) & \Rightarrow & a < c < b \\ & \Rightarrow & a^{n-1} < c^{n-1} < b^{n-1} \qquad ; a,b,c > 0 \\ & \Rightarrow & na^{n-1} < nc^{n-1} < nb^{n-1} \\ & \text{from (1)} \Rightarrow & n(b-a)a^{n-1} < b^n - a^n < n(b-a)b^{n-1} \end{array}.$$

Example 4: Find the intervals on which the curve $y = e^{-x^2}$ is convex and concave.

Solution: We have: $y^{'} = -2xe^{-x^2} \Rightarrow y^{''} = -2e^{-x^2} - 2x(-2x)e^{-x^2}$ and $y^{''} = 0 \Leftrightarrow e^{-x^2}(4x^2-2) = 0 \Rightarrow x_{1,2} = \pm\frac{1}{\sqrt{2}}$.

Since $y^{''} > 0$ for all $x \in \left(-\infty, \frac{-1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ then the curve is convex on this interval. On the other hand as $y^{''} < 0$ for all $x \in \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ then the curve is concave on this interval.

Example 5: Find the maksimum volume of the circular cylinder which can be inscribed in a sphere of radius a?

Solution: As we have
$(2r)^2 + h^2 = (2a)^2 \Rightarrow h^2 = 4\left(a^2 - r^2\right) \Rightarrow h = 2\sqrt{a^2 - r^2.}$
$V = \pi r^2 h = 2\pi r^2 \sqrt{a^2 - r^2}$
$V^{'}(r) = 4\pi r\sqrt{a^2 - r^2} - \frac{2\pi r^3}{\sqrt{a^2-r^2}} = 0 \Rightarrow r = \sqrt{\frac{2}{3}}a.$
$V"\left(\sqrt{\frac{2}{3}}a.\right) < 0$ we calculate the maximum volume as $V = \pi \frac{2}{3} a^3 2\sqrt{a^2 - \frac{2}{3}a^2} = \frac{4\sqrt{3}}{9}\pi a^3.$