4.1 Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the polar coordinate system, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

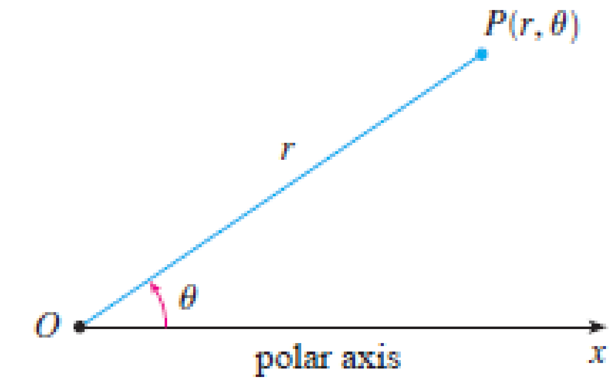


FIGURE 1

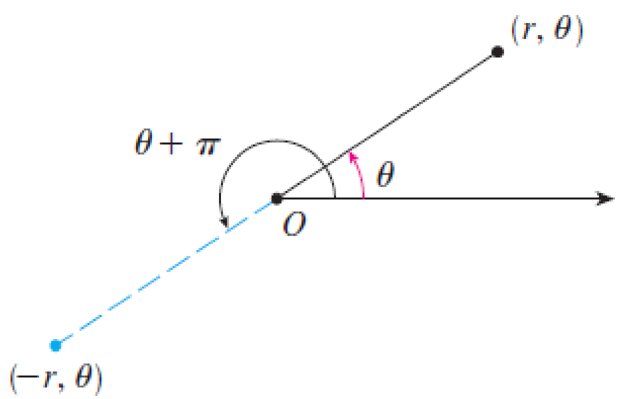


FIGURE 2

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. We use the convention that an angle r is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O , but on opposite sides of O.

If , $r > 0$ the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole.

Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Example 1 Plot the points whose polar coordinates are given.

$$(\boldsymbol{a})\ \left(1, \frac{5\pi}{4}\right), (\boldsymbol{b})\ (2, 3\pi), (\boldsymbol{c})\ \left(2, -\frac{2\pi}{3}\right), (\boldsymbol{d})\ \left(-3, \frac{3\pi}{4}\right)$$

Solution

The points are plotted in Figure 3.

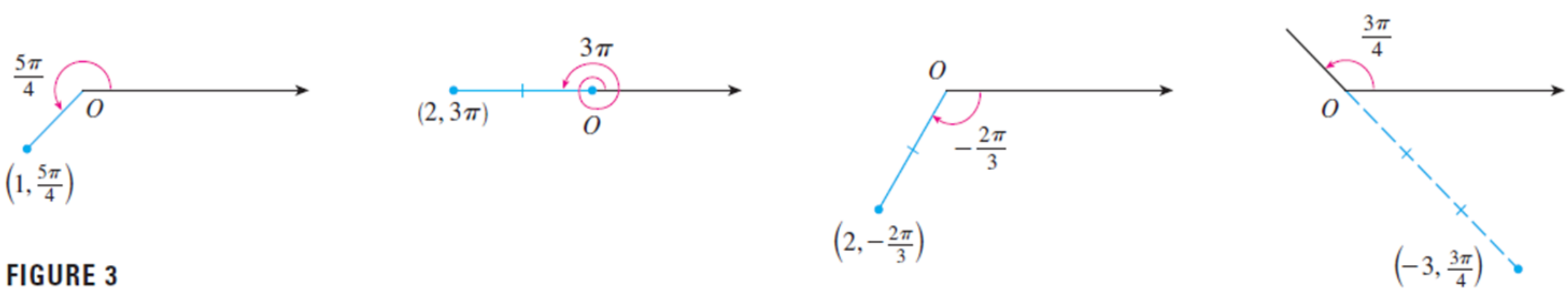


FIGURE 3

In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $\left(1, \frac{5\pi}{4}\right)$ in Example 1(a) can be written as $\left(1, \frac{-3\pi}{4}\right)$ or $\left(1, \frac{13\pi}{4}\right)$ or $\left(-1, \frac{\pi}{4}\right)$

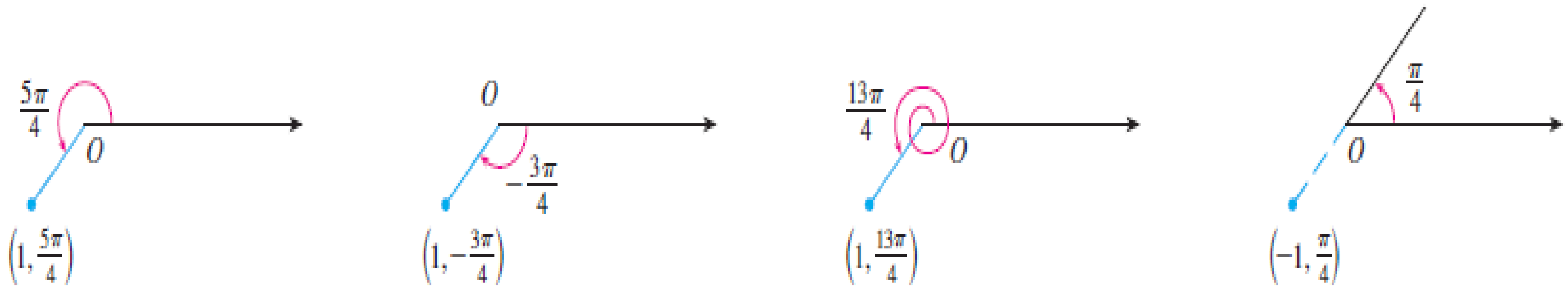


FIGURE 4

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n+1)\pi)$$

where n is any integer.

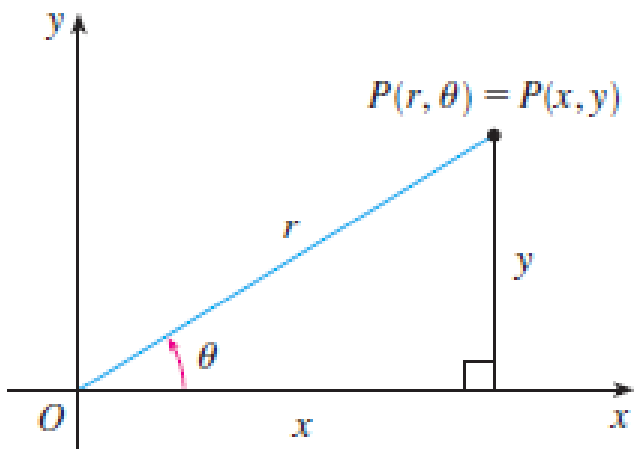


FIGURE 5

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure, we have

$$x = r\cos\theta\ ,\ y = r\sin\theta \quad \ldots(1)$$

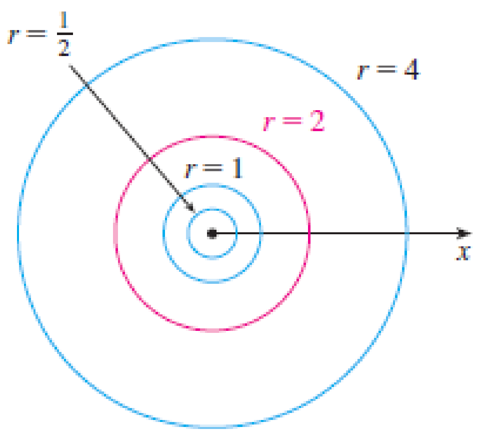
Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and θ when x and y are known, we use the equations

$$r^2 = x^2 + y^2 \, , \tan\theta = \frac{y}{x} \ldots (2)$$

Polar Curves

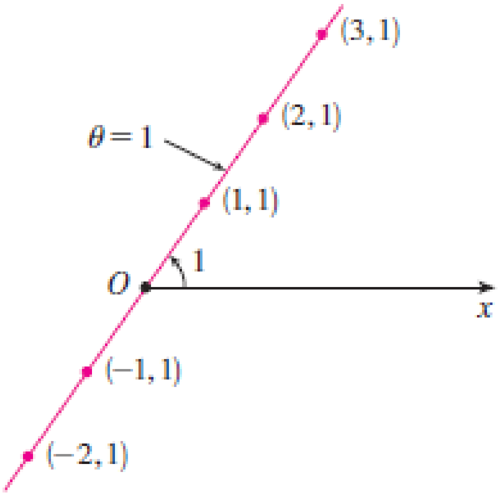
The graph of a polar equation $r = f(\theta)$, or more generally , $F(r, \theta)$ consists of all points P that have at least one polar representation (r,θ) whose coordinates satisfy the equation.

Example: What curve is represented by the polar equation $r = 2$?



The curve consists of all points (r,θ) with $r = 2$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$.

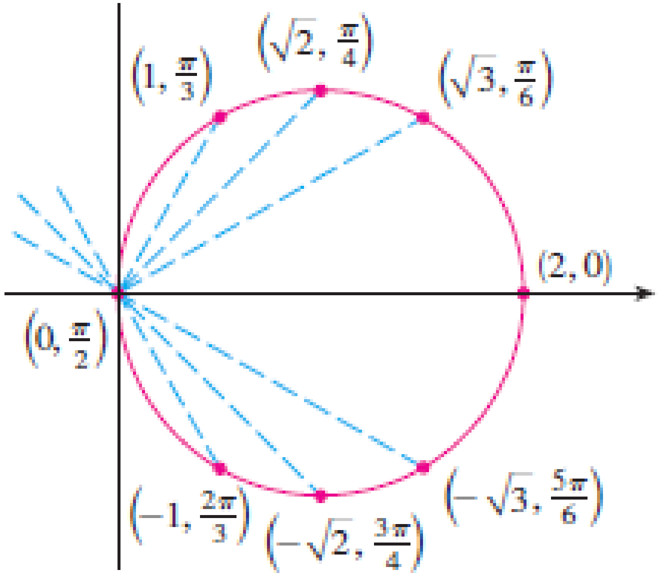
Example: Sketch the polar curve $\theta = 1$.



This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis as in the Figure. Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant.

Example: Sketch the polar curve $r = 2cos\theta$.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2



We find the values of r for some convenient values of θ and plot the corresponding points (r,θ). Then we join these points to sketch the curve, which appears to be a circle. We have used only values of θ between 0 and π , since if we let θ increase beyond π, we obtain the same points again.

Areas and Lengths in Polar Coordinates

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

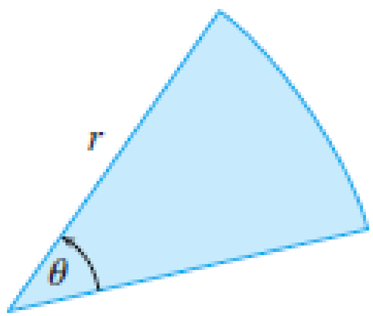


FIGURE 1

$$A = \frac{1}{2} r^2 \theta \ldots (1)$$

where, as in Fig 1 r is the radius and θ is the radian measure of the central angle.

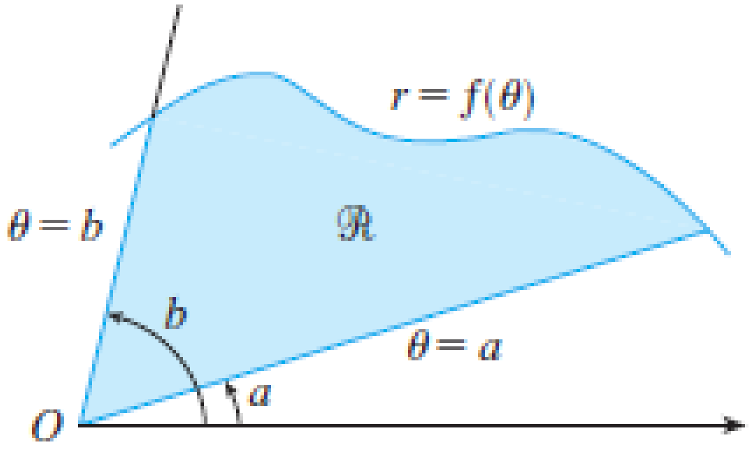


FIGURE 2

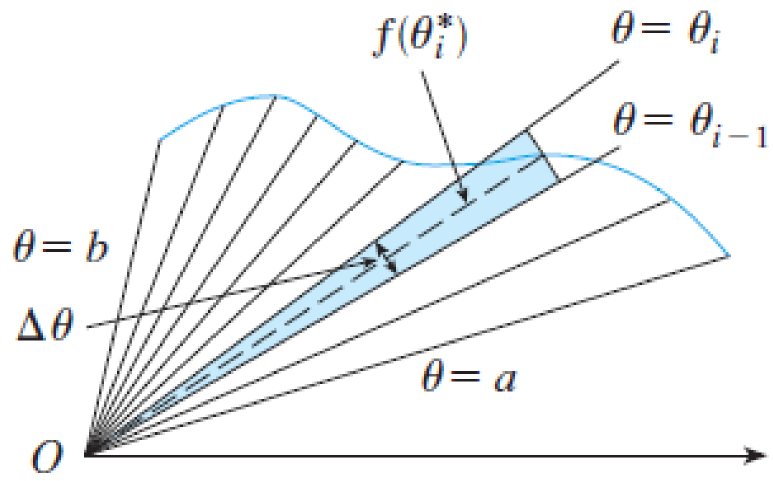


FIGURE 3

Example-1 Find the area enclosed by one loop of the four-leaved rose $r = cos2\theta$.

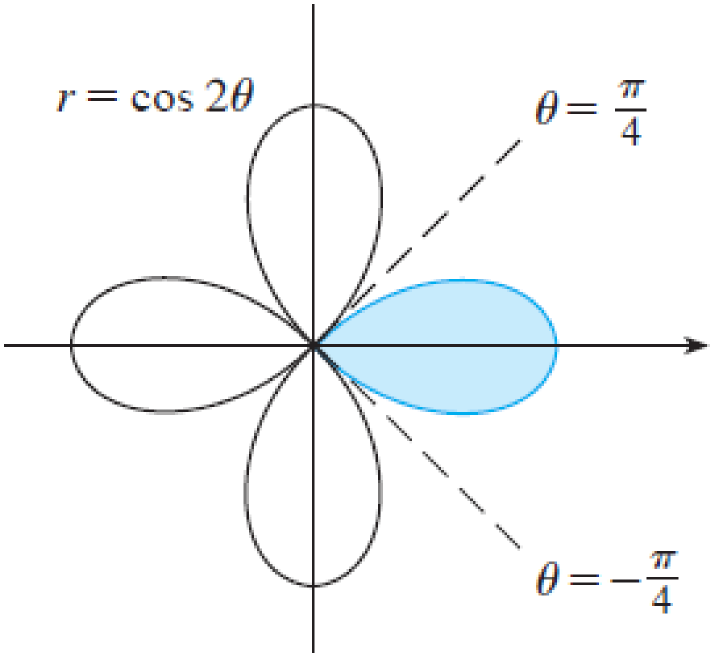


FIGURE 4