INTEGRALS

This chapter starts with the area problems and uses them to formulate the idea of a definite integral, which is the basic concept of integral calculus.

1.1 The Area Problem

We begin by attempting to solve the area problem:

Find the area of the region that lies under the curve

$$y = f(x)$$

from a to b.

This means that , illustrated in Figure 1, is bounded by the graph of a continuous function $f\ [where\ f(x) \geq 0], the\ vertical\ lines\ x = a\ and\ x = b, and\ the\ x - axis.$

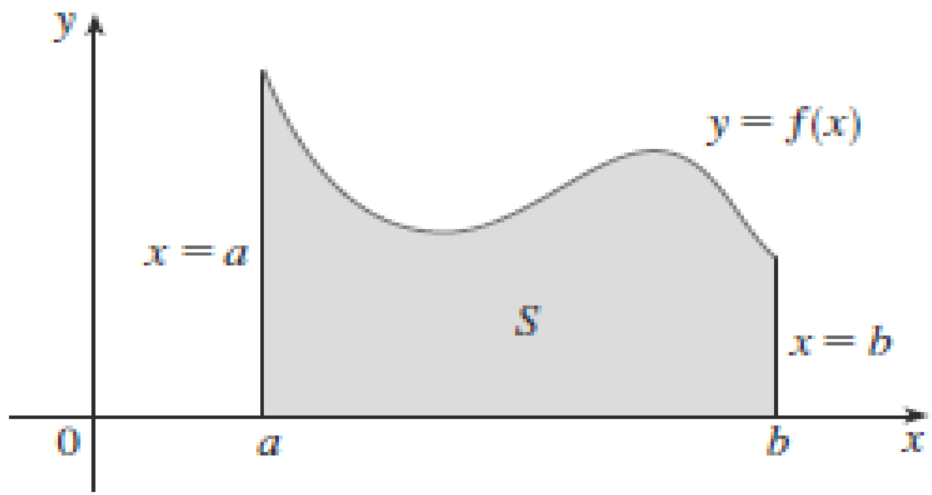


Figure 1

In trying to solve the area problem we have to ask ourselves: What is the meaning of the word **area?** This question is easy to answer for regions with **straight sides**.

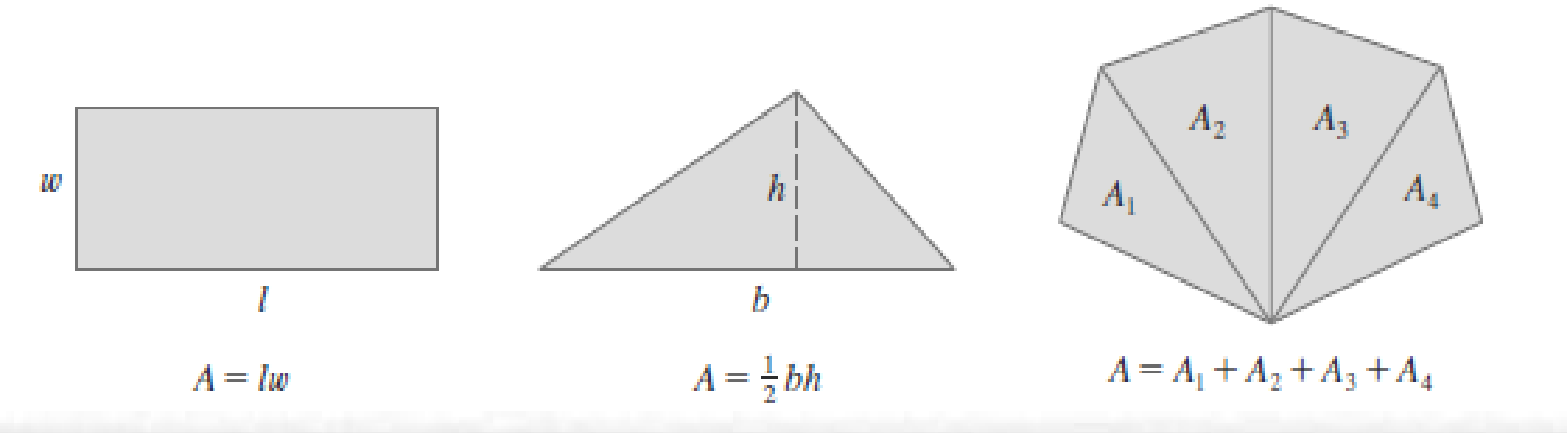


Figure 2

- For a rectangle, the area is defined as the product of the length and the width.

- The area of a triangle is half the base times the height.

- The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

However, it isn’t so easy to find the area of a region with **curved sides**. We all have an intuitive idea of what the area of a region is.

But part of the area problem is to make this intuitive idea precise by giving an **exact definition** of area.

We first approximate the region by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

- The following example illustrates the procedure.

Example-1

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

(the parabolic region S illustrated in Figure 3).

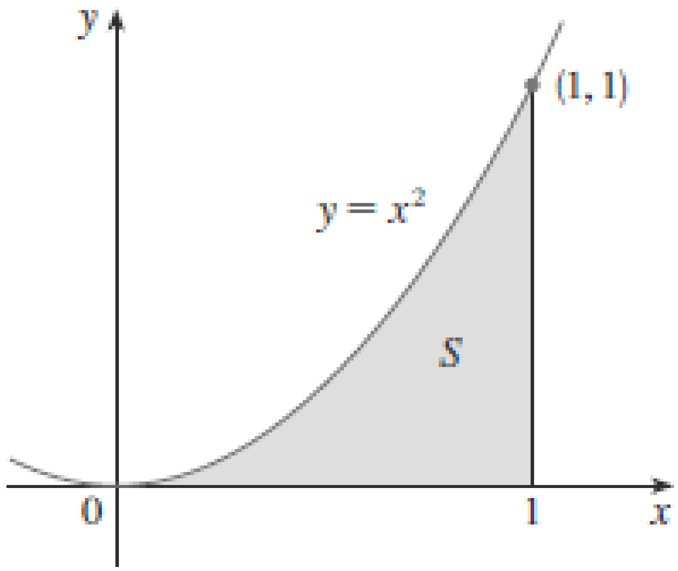
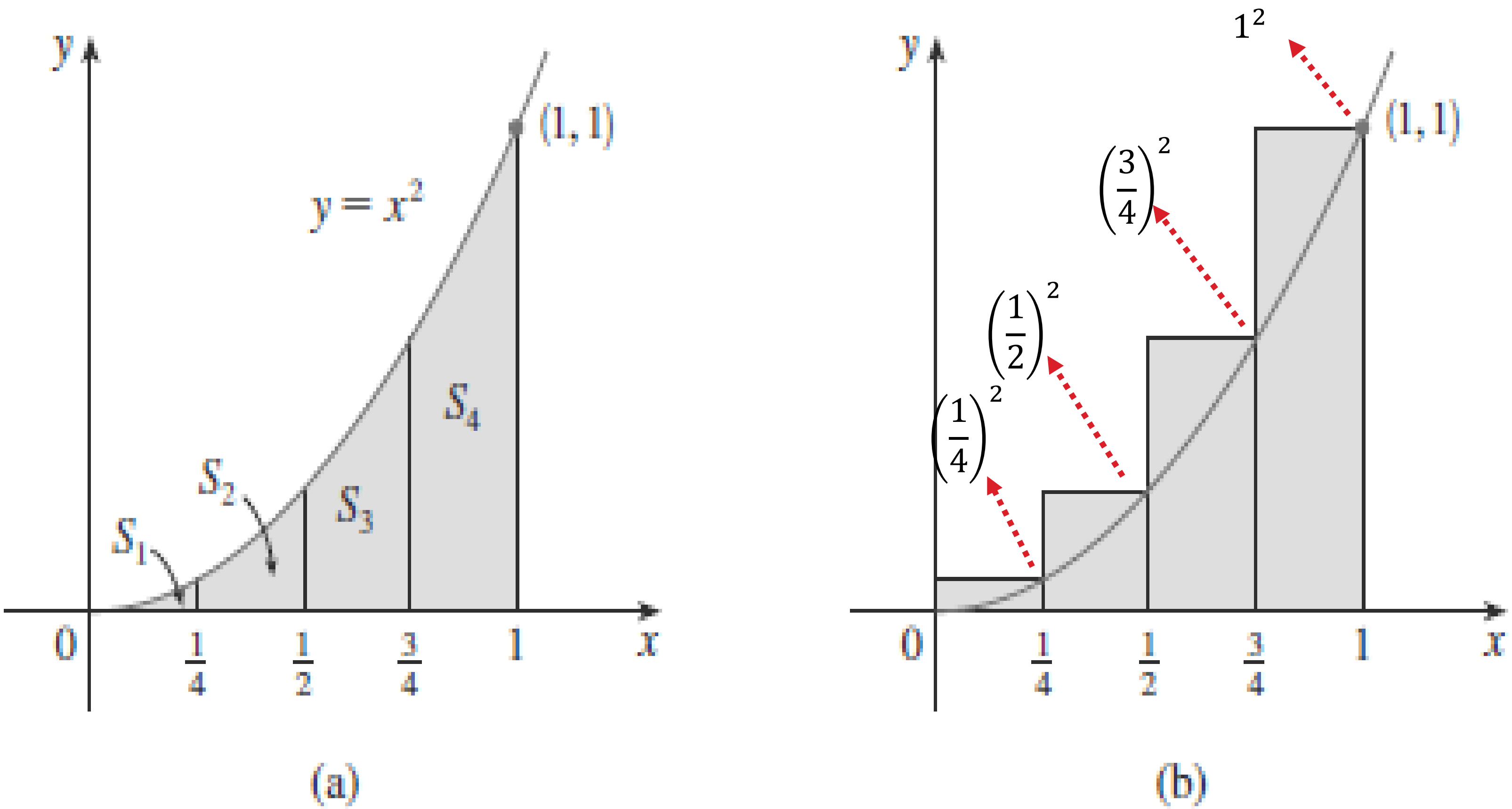


Figure 3

Solution

We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that.



y
x
0
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1
(1, 1)
$y = x^2$
S_1
S_2
S_3
S_4
(a)
1^2
$\left(\frac{3}{4}\right)^2$
$\left(\frac{1}{2}\right)^2$
$\left(\frac{1}{4}\right)^2$
(b)

Figure 4

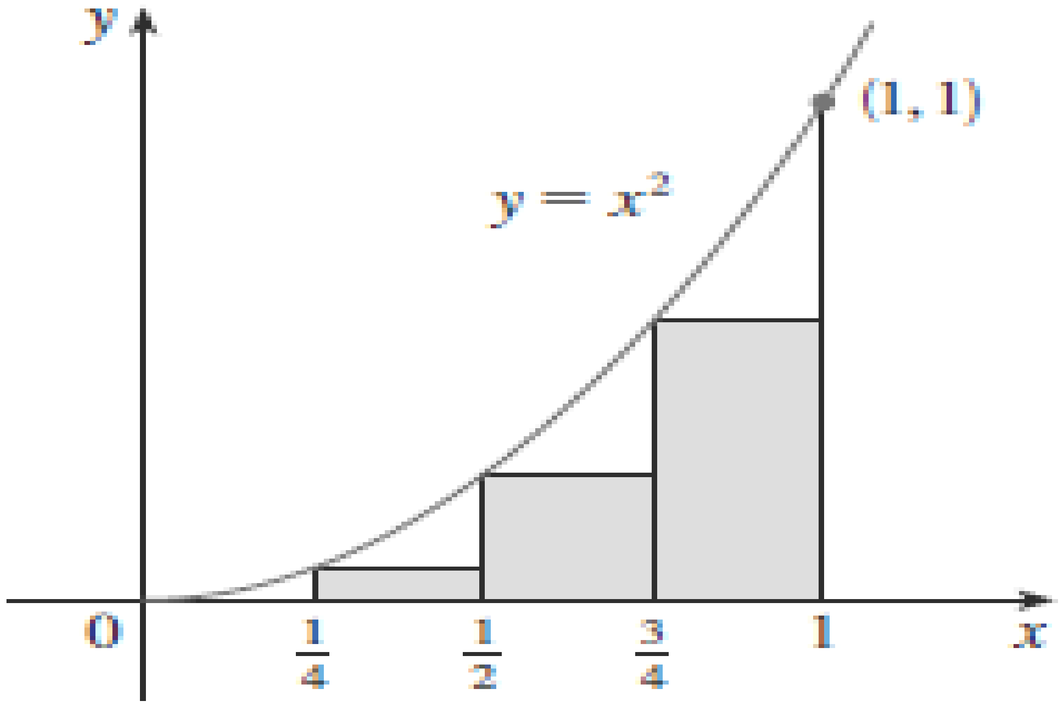


Figure 5

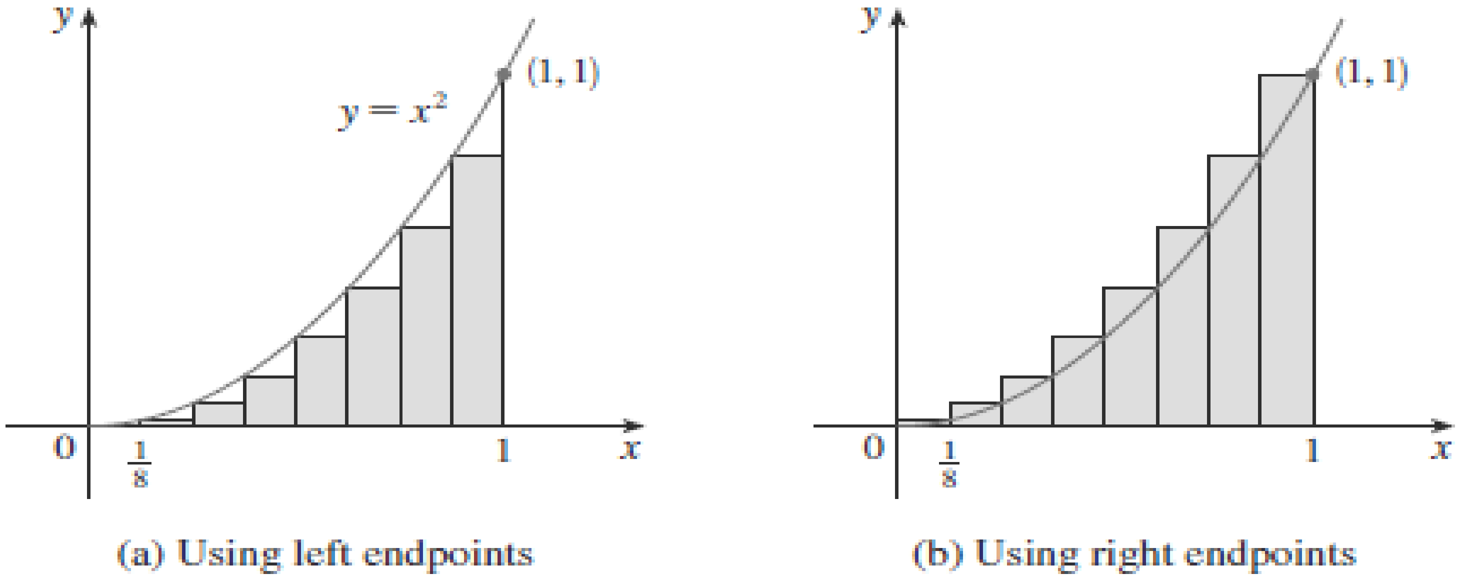

y
(1, 1)
y = x²
0
1/8
1
x
(a) Using left endpoints
y
(1, 1)
0
1/8
1
x
(b) Using right endpoints

Figure 6

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

From the values in the table in Example 1, it looks as if R_n is approaching $\frac{1}{3}$ as n increases.

We confirm this in the next example.