Example –3

Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

Solution

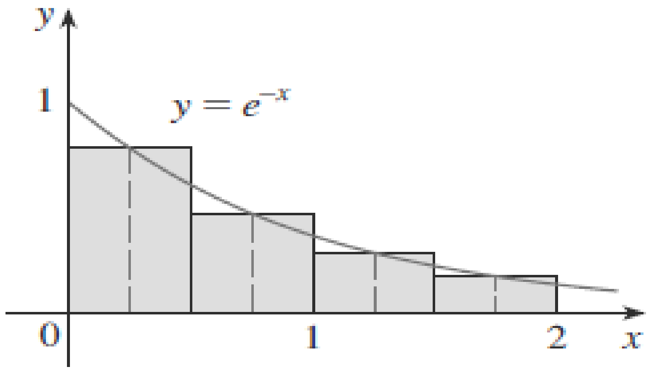

FIGURE 14

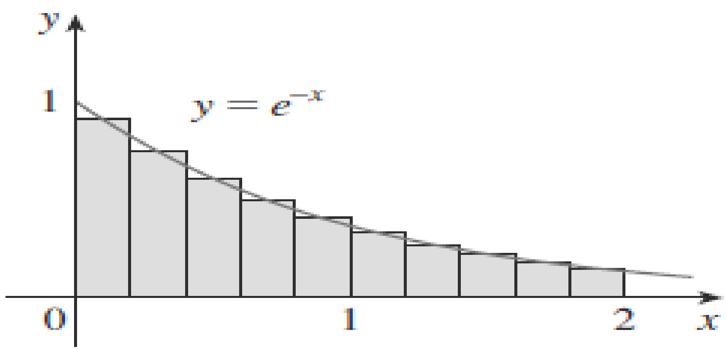

FIGURE 15

1.2 The Definite Integral

We saw in Section 1.1 that a limit of the form

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}[\,f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] \quad (1)$$

arises when we compute an area.

We also point out that limits of the form (1) also arise in finding

- lengths of curves,
- volumes of solids,
- centers of mass,
- force due to water pressure,

and work, as well as other quantities.

We therefore give this type of limit a special name and notation.

Definition 2 (Definite Integral)

If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0(= a)$, $x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad (2)$$

Because we have assumed that f is continuous, it can be proved that the limit in Definition 2 always exists and gives the same value no matter how we choose the sample points x_i^*.

If we take the sample points to be **right endpoints**, then $x_i^* = x_i$ and the definition of an integral becomes

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad (3)$$

If we choose the sample points to be **left endpoints**, then $x_i^* = x_{i-1}$ and the definition becomes

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

Alternatively, we could choose x_i^* to be the midpoint of the subinterval or any other number between x_{i-1} and x_i.

Although most of the functions that we encounter are continuous, the limit in Definition 2 also exists if has a finite number of removable or jump discontinuities (but not infinite discontinuities.) So we can also define the definite integral for such functions.

Note 1

The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is **the lower limit** and is b **the upper limit**. The symbol dx has no official meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The procedure of calculating an integral is called **integration**.

Note 2

The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x . In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

Note 3

The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). We know that if f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1). By comparing Definition 2 with the definition of area in Section 1.1, we see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b.

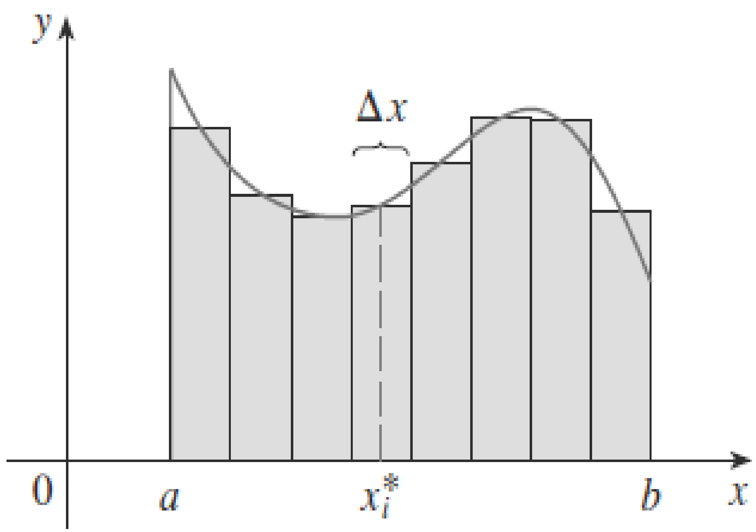

FIGURE 1

If $f(x) \geqslant 0$, the Riemann sum $\sum f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

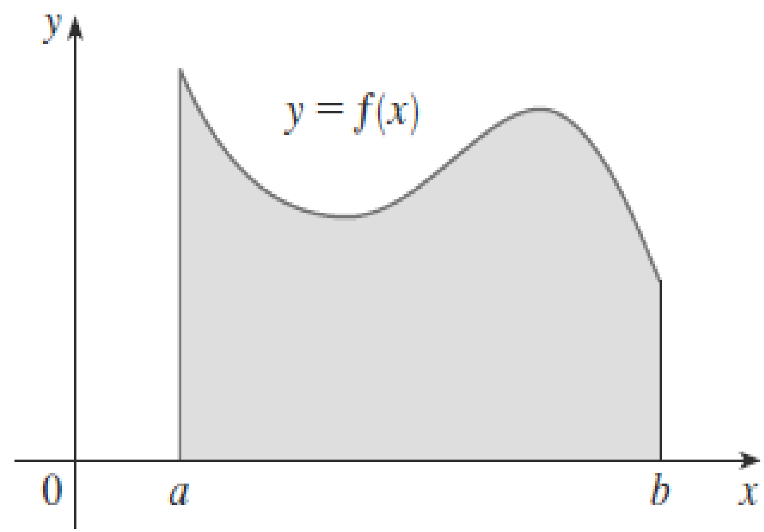

FIGURE 2

If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

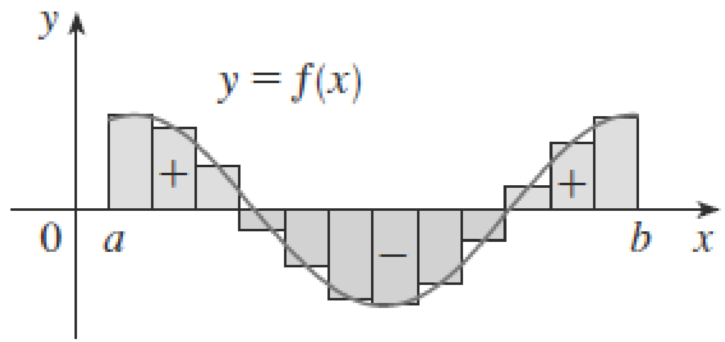

FIGURE 3
$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area

If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis.

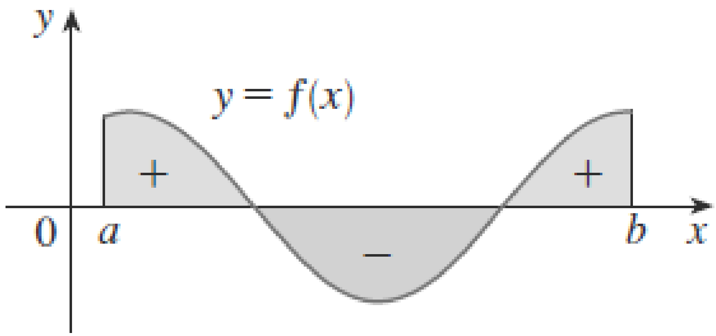

FIGURE 4
$\int_a^b f(x)\,dx$ is the net area

When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

Note 4

In the spirit of the precise definition of the limit of a function, we can write the precise meaning of the limit that defines the integral in Definition 2 as follows:

For every number $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

Note 5

Although we have defined $\int_a^b f(x)\,dx$ by dividing $[a, b]$ into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width. And there are methods for numerical integration that take advantage of unequal subintervals.

If the subinterval widths are Δx_1, Δx_2, ..., Δx_n we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x$$