1.3. The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: **differential calculus and integral calculus.**

Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's teacher at Cambridge, Isaac Barrow (1630–1677), discovered that these two problems are actually closely related. In fact, he realized that **differentiation and integration are inverse processes.**

The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibniz who exploited this relationship and used it to develop calculus into a systematic mathematical method.

In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums as we did in Sections 1.1 and 1.2.

The first part of the Fundamental Theorem deals with functions defined by an equation of the form

$$g(x) = \int_a^x f(t)dt \qquad (1)$$

where f is a continuous function on $[a, b]$ and x varies between a and b. Observe that g depends only on x, which appears as the variable upper limit in the integral.

If x is a fixed number, then the integral $\int_a^x f(t)dt$ is a definite number. If we then let x vary, the number $\int_a^x f(t)dt$ also varies and defines a function of x denoted by $g(x)$.

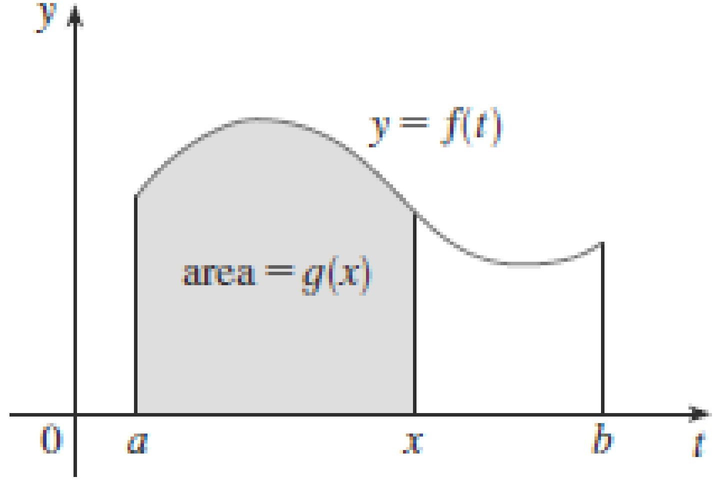

FIGURE 1

If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. (Think of as the "area so far" function; see Figure 1.)

The Fundamental Theorem of Calculus Part 1 (FTC1)

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt\,, a \leq x \leq b$$

is continuous on $[a, b]$, and differentiable on (a, b), and $g'(x) = f(x)$.

- This theorem says that the derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit.

Example-1

Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\,dt$.

Solution

Since $f(t) = \sqrt{1 + t^2}$ is continuous by using FTC1,

$$g'(x) = \sqrt{1 + x^2}$$

Example-2

Although a formula of the form $g(x) = \int_a^x f(t)dt$ may seem like a strange way of defining a function, books on physics, chemistry, and statistics are full of such functions.

For instance, **the Fresnel function**

$$S(x) = \int_0^x \sin\left({\pi t^2}/_2\right)dt$$

is named after the French physicist Augustin Fresnel (1788–1827), who is famous for his works in optics. This function first appeared in Fresnel's theory of the diffraction of light waves, but more recently it has been applied to the design of highways.

Part 1 of the Fundamental Theorem tells us how to differentiate the Fresnel function:

$$S'(x) = \sin({}^{\pi x^2}/_2)$$

This means that we can apply all the methods of differential calculus to analyze S.

Figure 2 shows the graphs of $f(x) = \sin({}^{\pi x^2}/_2)$ and the Fresnel function $S(x) = \int_0^x \sin({}^{\pi t^2}/_2)dt$

Figure 3 shows a larger part of the graph of .

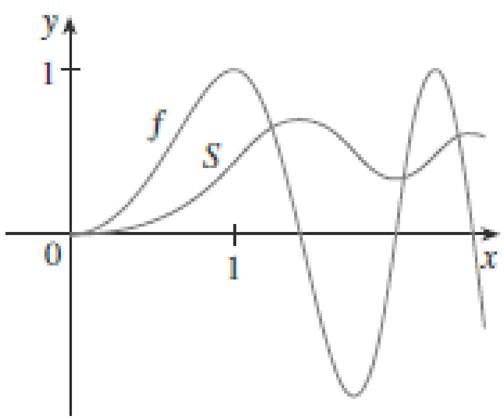

Figure-2

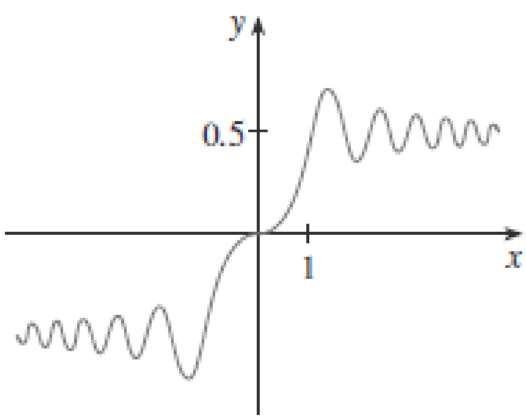

Figure-3

In Section 1.2 we computed integrals from the definition as a limit of Riemann sums and we saw that this procedure is sometimes long and difficult.

The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

The Fundamental Theorem of Calculus Part 2 (FTC2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Part 2 of the Fundamental Theorem states that if we know an antiderivative F of f, then we can evaluate $\int_a^b f(x)dx$ simply by subtracting the values of F at the endpoints of the interval $[a, b]$. It's very surprising that $\int_a^b f(x)dx$, which was defined by a complicated procedure involving all of the values of $f(x)$ for $a \leq x \leq b$, can be found by knowing the values of $F(x)$ at only two points, a and b.

Example-3

Evaluate the integral $\int_1^3 e^x \mathrm{dx}$.

Solution

The function $f(x) = e^x$ is continuous everywhere and we know that an antiderivative is $F(x) = e^x$, so Part 2 of the Fundamental Theorem gives

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e$$

Notice that FTC2 says we can use any antiderivative F of f. So we may as well use the simplest one, namely $F(x) = e^x$ instead of $e^x + 7\ or\ e^x + C$.

Note: We often use the notation

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F' = f$.

Example-4

Find the area under the parabola $y = x^2$ from 0 to 1.

OR

Evaluate the integral $\int_0^1 x^2 dx$.

Example-5

Evaluate $\int_3^6 \frac{dx}{x}$.

Example-6

Find the area under the cosine curve from 0 to b , where $0 \leq b \leq \frac{\pi}{2}$.

Example-7

What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2} dx = \frac{x^{-1}}{-1}\Bigg|_{-1}^{3} = \frac{-1}{3} - 1 = \frac{-4}{3}$$

The Fundamental Theorem of Calculus say that **differentiation and integration are inverse processes.** Each undoes what the other does.

The Fundamental Theorem of Calculus is unquestionably the most important theorem in calculus and, indeed, it ranks as one of the great accomplishments of the human mind. Before it was discovered, from the time of Eudoxus and Archimedes to the time of Galileo and Fermat, **problems of finding areas, volumes, and lengths of curves were so difficult** that only a genius could meet the challenge. But now, armed with the systematic method that Newton and Leibniz fashioned out of the Fundamental Theorem, we will see in the chapters to come that these challenging problems are accessible to all of us.

Theorem (Symmetry Rule)

Let f be a continuous function defined on $[-a, a]$

(i) If f is an **odd** function, then $\int_{-a}^{a} f(x)dx = 0$

(ii) If f is an **even** function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

Substitution Rule for Definite Integral

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

This rule says that when using a substitution in a definite integral, we must put everything in terms of the new variable , not only and but also the limits of integration.

Example-8

Evaluate the following integrals:

a) $\int_0^4 \sqrt{2x+1}\,dx$

b) $\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

c) $\int_0^{\sqrt{\pi}} x\cos x^2\,dx$

Solution

a) $\int_0^4 \sqrt{2x+1}\, dx$

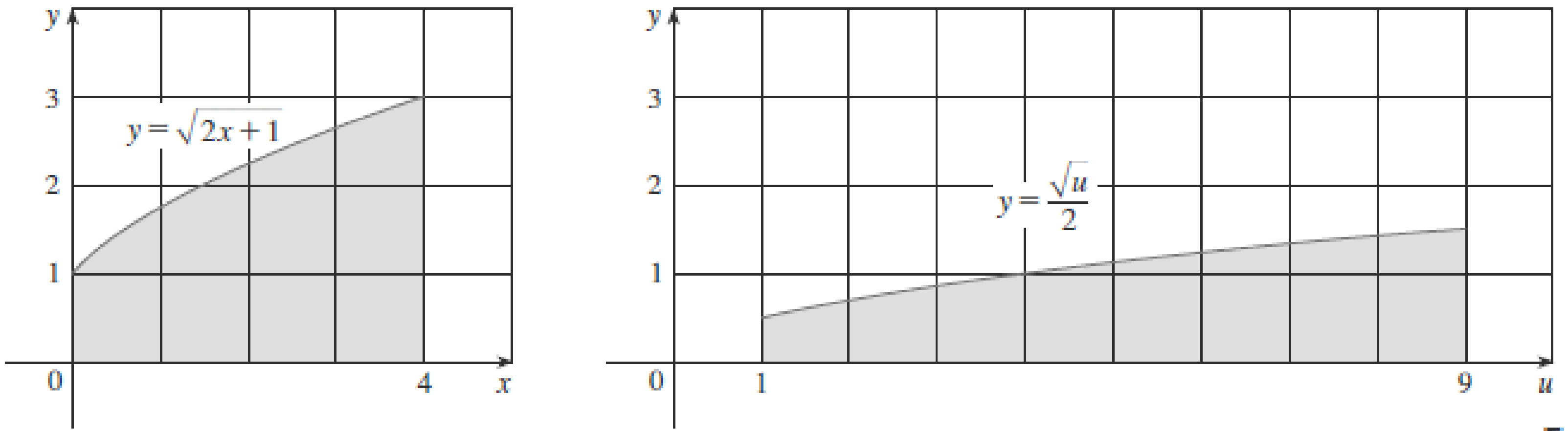

FIGURE 2

The geometric interpretation of Example (8.a) is shown in Figure 2. The substitution $u = 2x + 1$stretches the interval [0,4] by a factor of 2 and translates it to the right by unit. The Substitution Rule shows that the two areas are equal.

More examples will be solved in the class.

Average Value of a Function

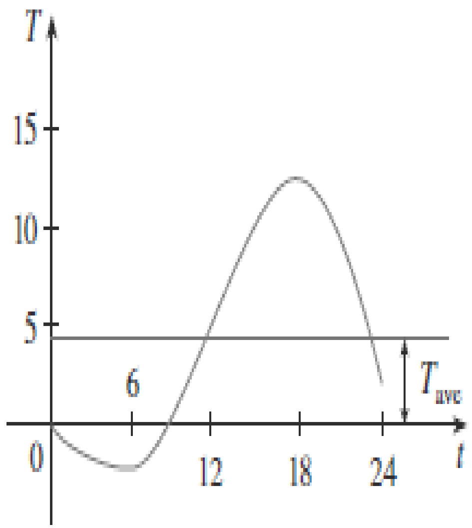

FIGURE 1

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{ave} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible? Figure 1 shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{ave}.

In general, let's try to compute the average value of a function $y = f(x)$, $a \leq x \leq b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$. Then we choose points $x_1^*, \ldots, x_n^*$ in successive subintervals and calculate the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

(For example, if f represents a temperature function and $n = 24$, this means that we take temperature readings every hour and then average them.) Since $\Delta x = (b - a)/n$, we can write $n = (b - a)/\Delta x$ and the average value becomes

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} = \frac{1}{b-a}[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

If we let n increase, we would be computing the average value of a large number of closely spaced values. (For example, we would be averaging temperature readings taken every minute or even every second.) The limiting value is

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

by the definition of a definite integral.

Therefore, we define the **average value of f** on the interval $[a, b]$ as

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example

Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1,2]$.

Solution

With $a = -1$ and $b = 2$, we have

$$f_{ave} = \frac{1}{(2-(-1))}\int_{-1}^{2}(1+x^2)dx = 2$$

The question arises: Is there a number c at which the value of f is exactly equal to the average value of the function, that is, $f(c) = f_{ave}$? The following theorem says that this is true for continuous functions.

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

EXAMPLE 2 Since $f(x) = 1 + x^2$ is continuous on the interval $[-1, 2]$, the Mean Value Theorem for Integrals says there is a number c in $[-1, 2]$ such that

$$\int_{-1}^{2} (1 + x^2)\,dx = f(c)[2 - (-1)]$$

In this particular case we can find c explicitly. From Example 1 we know that $f_{\text{ave}} = 2$, so the value of c satisfies

$$f(c) = f_{\text{ave}} = 2$$

Therefore $\qquad 1 + c^2 = 2 \qquad \text{so} \qquad c^2 = 1$

Thus, in this case there happen to be two numbers $c = \pm 1$ in the interval $[-1, 2]$ that work in the Mean Value Theorem for Integrals.