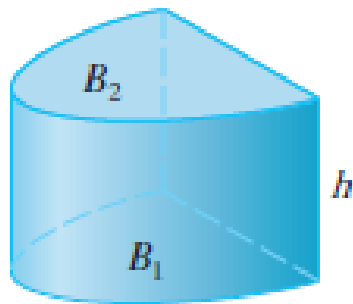


## 2.2 Volumes

In trying to find the volume of a solid we face the same type of problem as in finding areas. We have an intuitive idea of what volume means, but we must make this idea precise by using calculus to give an exact definition of volume.



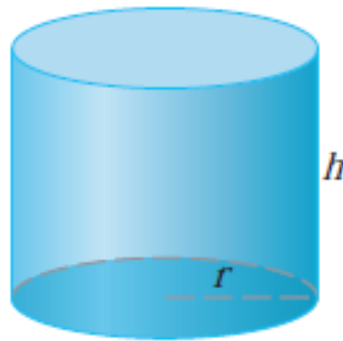
(a) Cylinder  $V = Ah$

Figure 1(a)

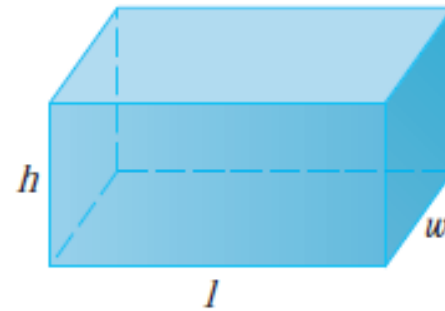
We start with a simple type of solid called a **cylinder** (or, more precisely, a right cylinder). As illustrated in Figure 1(a), a cylinder is bounded by a plane region  $B_1$ , called the **base**, and a congruent region  $B_2$  in a parallel plane. The cylinder consists of all points on line segments that are perpendicular to the base and join  $B_1$  to  $B_2$ .

If the area of the base is  $A$  and the height of the cylinder (the distance from  $B_1$  to  $B_2$ ) is  $h$ , then the volume  $V$  of the cylinder is defined as

$$V = Ah$$



(b) Circular cylinder  $V = \pi r^2 h$



(c) Rectangular box  $V = lwh$

Figure 1 (b)-(c)

In particular, if the base is a circle with radius  $r$ , then the cylinder is a circular cylinder with volume  $V = \pi r^2 h$  [see Figure 1(b)], and if the base is a rectangle with length  $l$  and width  $w$ , then the cylinder is a rectangular box (also called a rectangular parallelepiped) with volume  $V = lwh$  [see Figure 1(c)].

For a solid  $S$  that isn't a cylinder we first “**cut**”  $S$  into pieces and approximate each piece by a cylinder. We estimate the volume of  $S$  by adding the volumes of the cylinders. We arrive at the exact volume of  $S$  through a limiting process in which the number of pieces becomes large.

We start by intersecting  $S$  with a plane and obtaining a plane region that is called a **cross-section of  $S$** . Let  $A(x)$  be the area of the cross-section of  $S$  in a plane  $P_x$  perpendicular to the  $x$ -axis and passing through the point  $x$ , where  $a \leq x \leq b$ .

(See Figure 2. Think of slicing  $S$  with a knife through  $x$  and computing the area of this slice.) The cross-sectional  $A(x)$  area will vary as  $x$  increases from  $a$  to  $b$ .

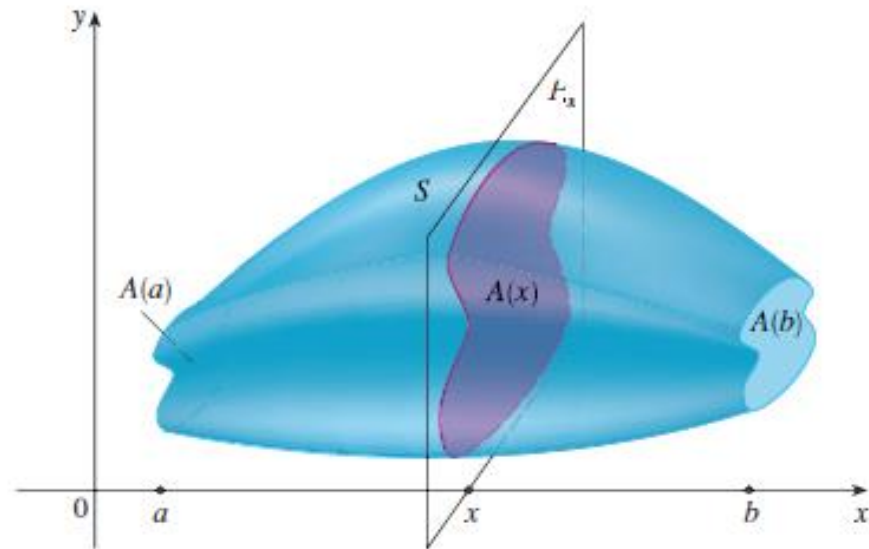


FIGURE 2

Let's divide  $S$  into  $n$  "slabs" of equal width  $\Delta x$  by using the planes  $P_{x_1}, P_{x_2}, \dots$  to slice the solid. (Think of slicing a loaf of bread.) If we choose sample points  $x_i^*$  in  $[x_{i-1}, x_i]$ , we can approximate the  $i$ th slab  $S_i$  (the part of  $S$  that lies between the planes  $P_{x_{i-1}}$  and  $P_{x_i}$ ) by a cylinder with base area  $A(x_i^*)$  and "height"  $\Delta x$ . (See Figure 3.)

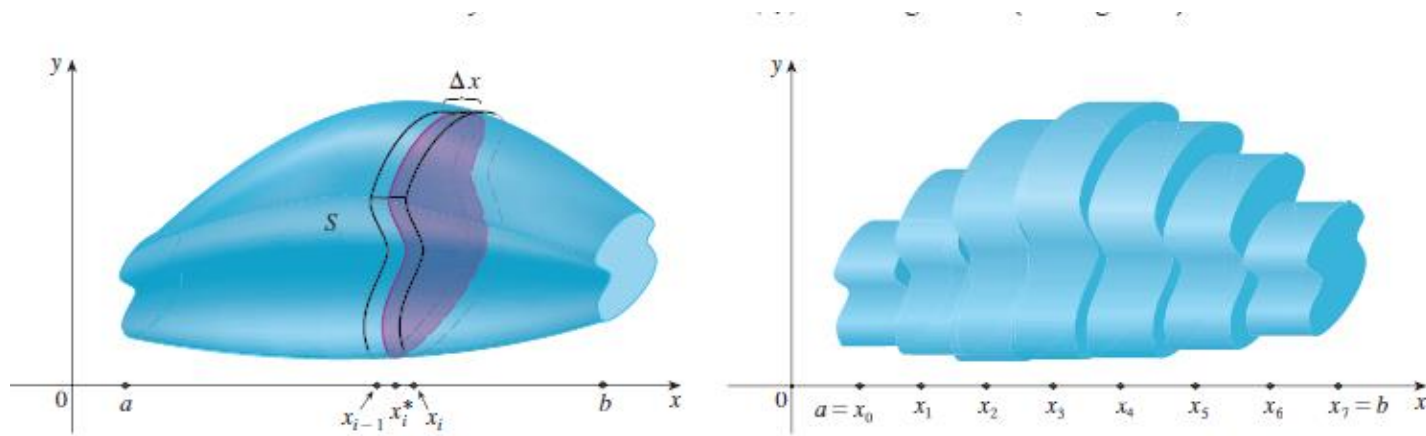


FIGURE 3

The volume of this cylinder is  $A(x_i^*)\Delta x$ , so an approximation to our intuitive conception of the volume of the  $i$ th slab  $S_i$  is

$$V(S_i) \approx A(x_i^*)\Delta x$$

Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^n A(x_i^*)\Delta x$$

This approximation appears to become better and better as  $n \rightarrow \infty$ . (Think of the slices as becoming thinner and thinner.) Therefore we **define** the volume as the limit of these sums as  $n \rightarrow \infty$ . But we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

## Definition of Volume

Let  $S$  be a solid that lies between  $x = a$  and  $x = b$ . If the cross-sectional area of  $S$  in the plane  $P_x$ , through  $x$  and perpendicular to the  $x$  - *axis*, is  $A(x)$ , where  $A$  is a continuous function, then the **volume** of  $S$  is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*) \Delta x = \int_a^b A(x) dx$$

When we use the volume formula  $V = \int_a^b A(x) dx$ , it is important to remember that  $A(x)$  is the area of a moving cross-section obtained by slicing through  $x$  perpendicular to the  $x$  - *axis*.

- It can be proved that this definition is independent of how  $S$  is situated with respect to the  $x$  - *axis*. In other words, no matter how we slice  $S$  with parallel planes, we always get the same answer for  $V$ .

$V = \int_a^b A(x)dx$  is known as Cavalieri's principle. This principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume.



## Example -1

Show that the volume of a sphere of radius  $r$  is  $V = \frac{4}{3}\pi r^3$ .

## Solution

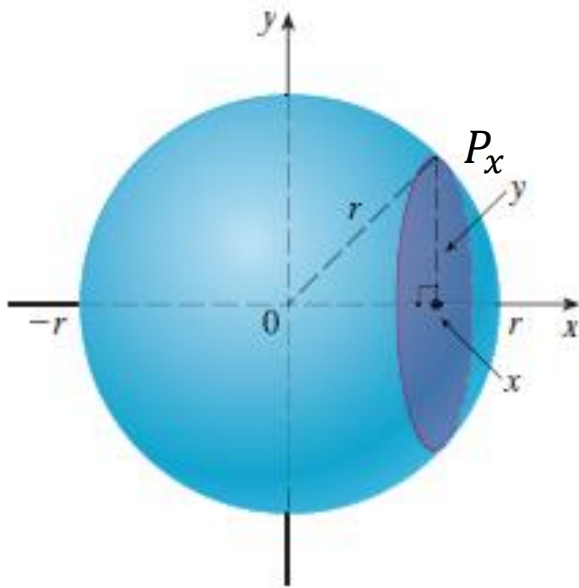


FIGURE 4

We have two more examples!!!

## Example -2

Find the volume of the solid obtained by rotating about the x-axis the region under the curve  $y = \sqrt{x}$  from 0 to 1.

## Solution

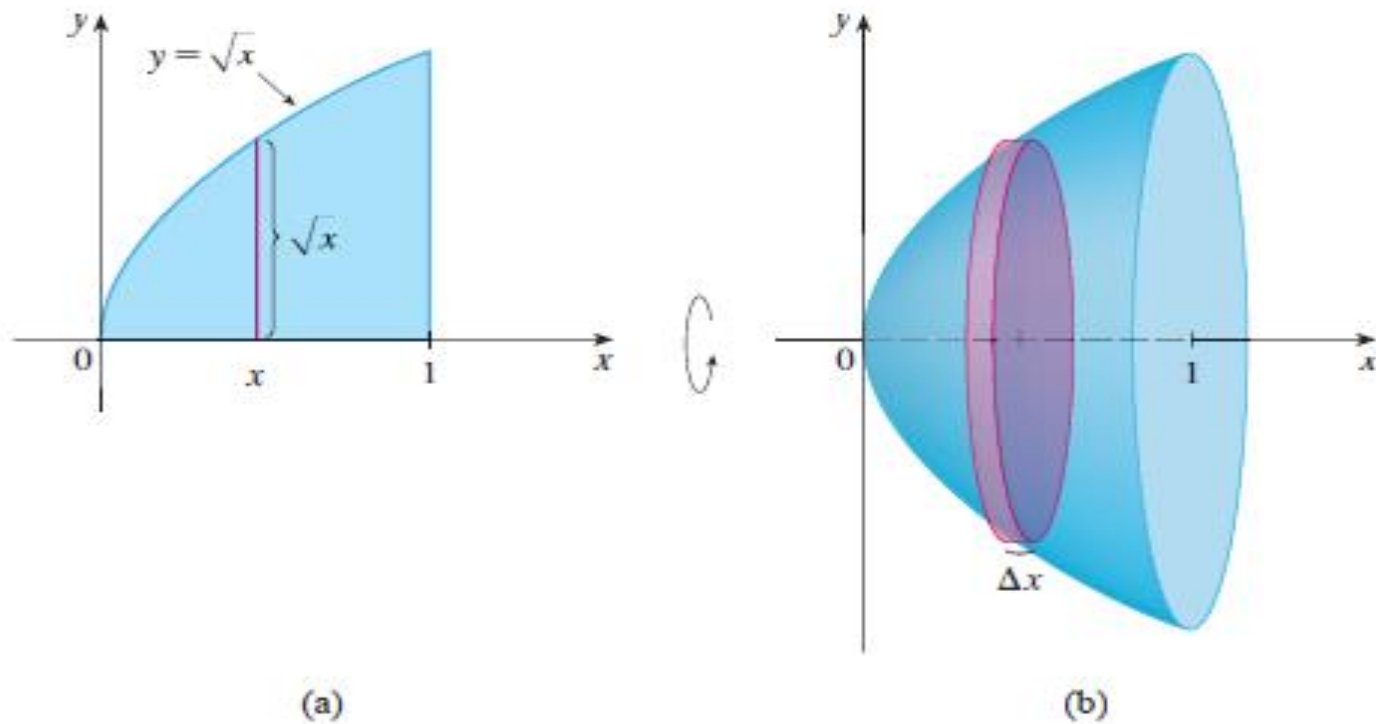


Figure 5

### Example -3

Find the volume of the solid obtained by rotating the region bounded by  $y = x^3$ ,  $y = 8$ , and  $x = 0$  about the  $y$ -axis.

### Solution

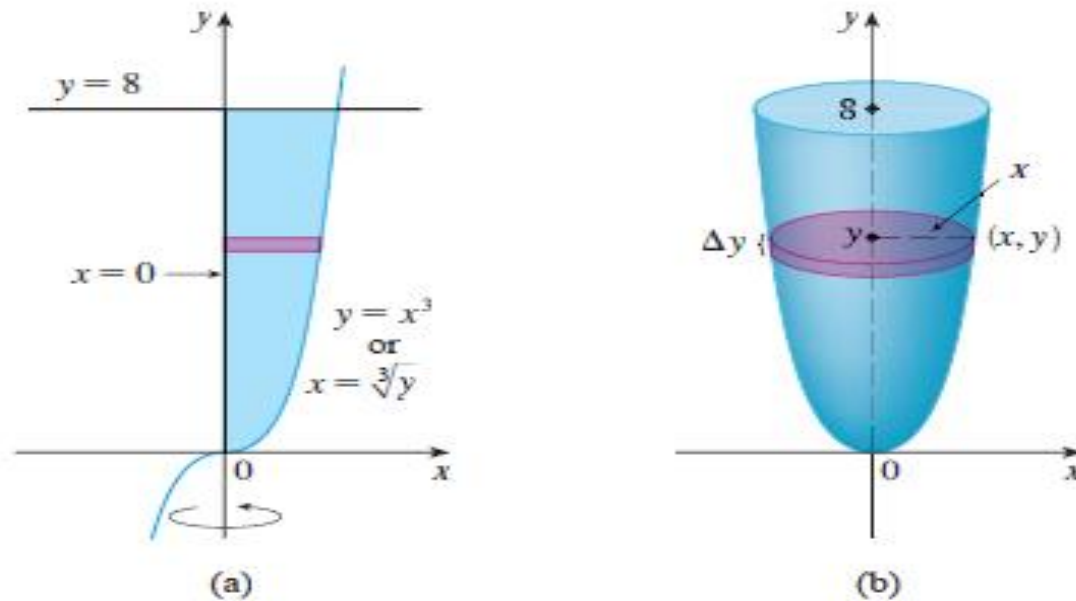


Figure 6

## Example -4

The region  $R$  enclosed by the curves  $y = x$  and  $y = x^2$  is rotated about the  $x$ -axis. Find the volume of the resulting solid.

## Solution

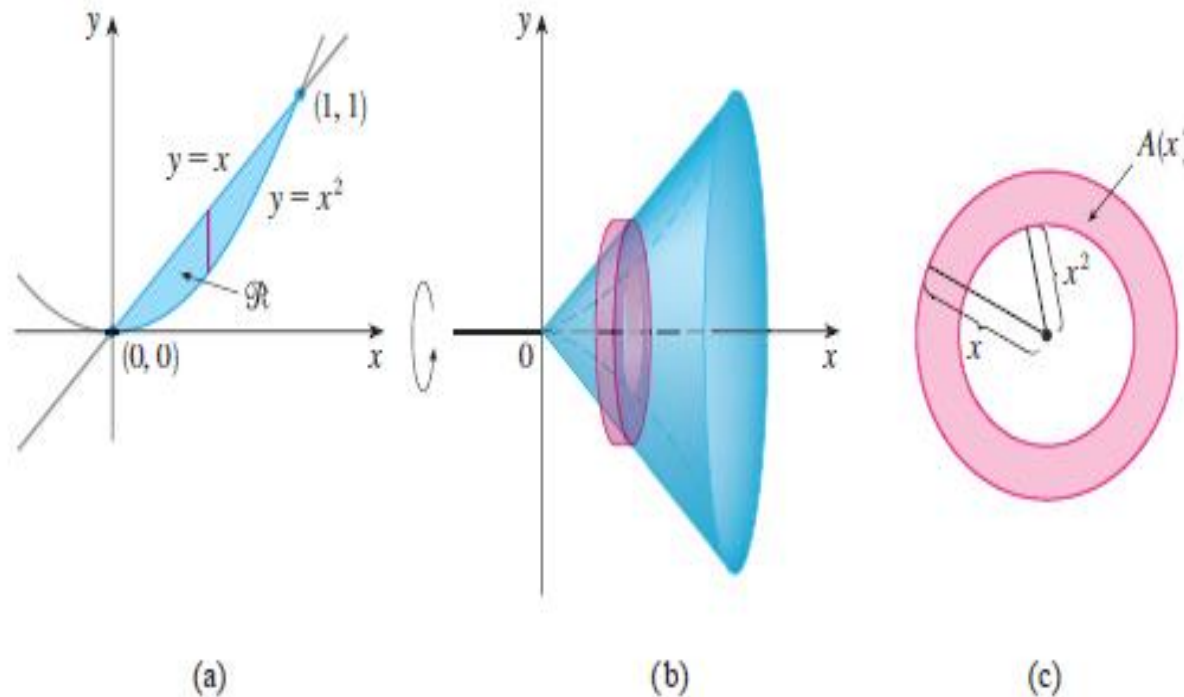


Figure 7

## Example -5

Find the volume of the solid obtained by rotating the region in Example 4 about the line  $y = 2$ .

## Solution

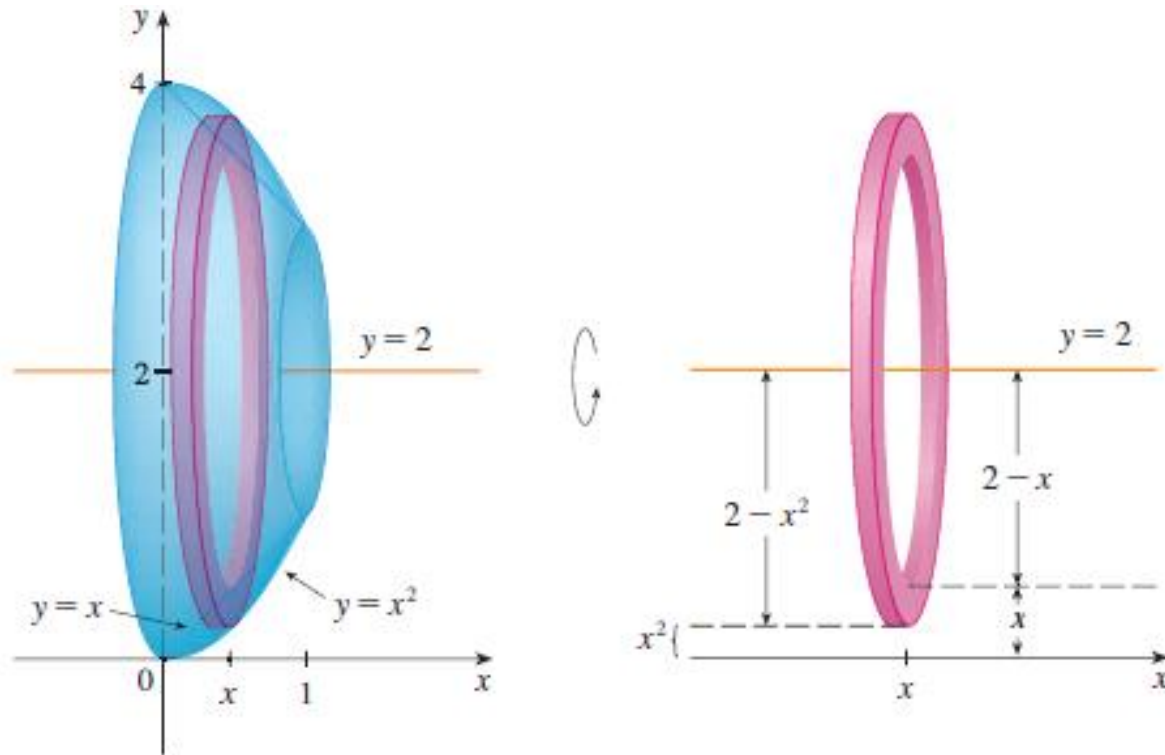


Figure 8

The solids in Examples 2–6 are all called **solids of revolution** because they are obtained by revolving a region about a line. In general, we calculate the volume of a solid of revolution by using the basic defining formula

$$V = \int_a^b A(x)dx \quad \text{or} \quad V = \int_c^d A(y)dy$$

and we find the cross-sectional area  $A(x)$  or  $A(y)$  in one of the following ways:

- ▶ If the cross-section is a disk (as in Examples 1–3), we find the radius of the disk (in terms of  $x$  or  $y$ ) and use

$$A = \pi(\text{radius})^2$$

- ▶ If the cross-section is a washer (as in Examples 4 and 5), we find the inner radius  $r_{in}$  and outer radius  $r_{out}$  from a sketch (as in Figures 7, 8 and 9) and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

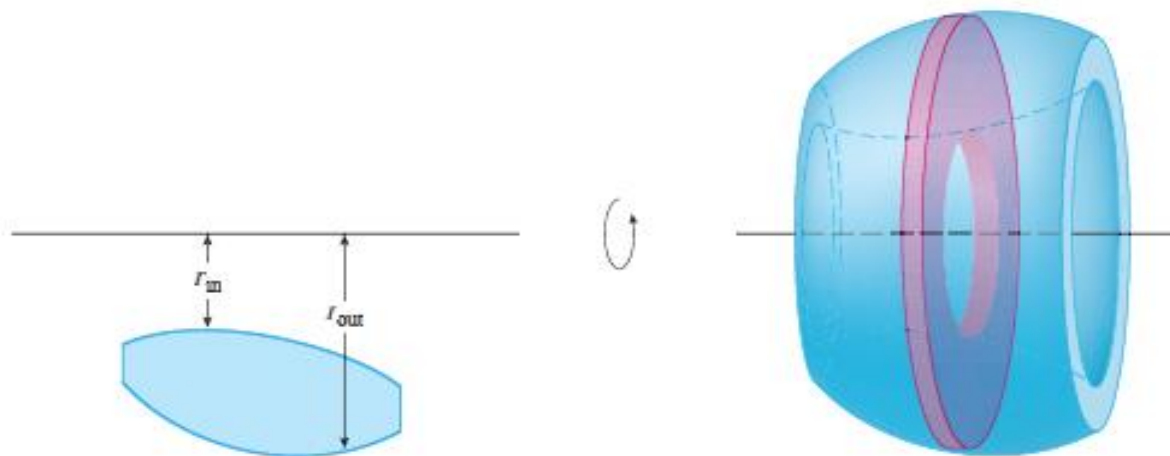


Figure 9

## Example -6

Find the volume of the solid obtained by rotating the region in Example 4 about the line  $x = -1$ .

## Solution

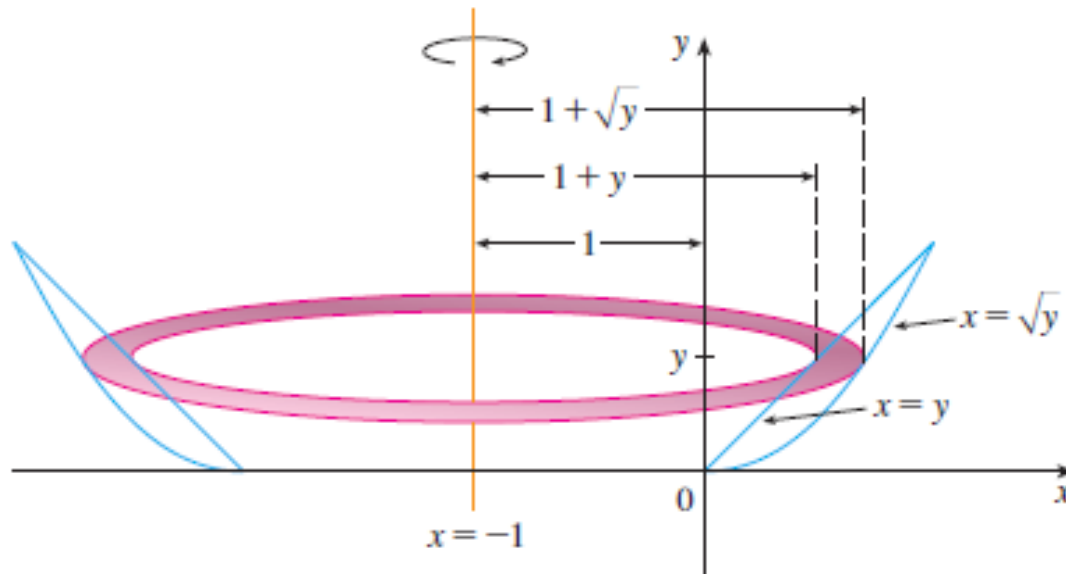


Figure 10



## 2.3 Volumes by Cylindrical Shells

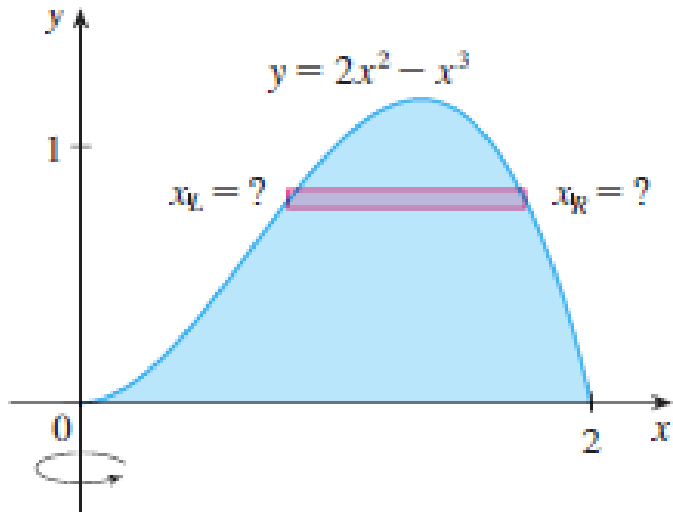


FIGURE 1

Some volume problems are very difficult to handle by the methods of the preceding section. For instance, let's consider the problem of finding the volume of the solid obtained by rotating about the  $y$ -axis the region bounded by  $y = 2x^2 - x^3$  and  $y = 0$ . (See Figure 1) If we slice perpendicular to the  $y$ -axis, we get a washer. But to compute the inner radius and the outer radius of the washer, we'd have to solve the cubic equation  $y = 2x^2 - x^3$  for  $x$  in terms of  $y$ , that's not easy.

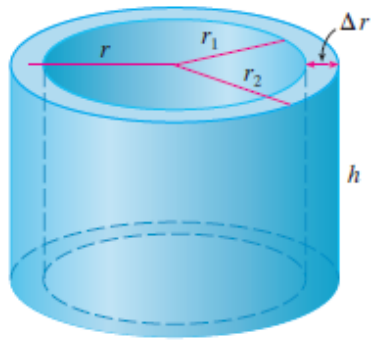


FIGURE 2

Fortunately, there is a method, called the **method of cylindrical shells**, that is easier to use in such a case. Figure 2 shows a cylindrical shell with inner radius  $r_1$ , outer radius  $r_2$  and height  $h$ . Its volume  $V$  is calculated by subtracting the volume  $V_1$  of the inner cylinder from the volume  $V_2$  of the outer cylinder

$$\begin{aligned}
 V &= V_2 - V_1 \\
 &= \pi r_2^2 h - \pi r_1^2 h \\
 &= \pi(r_2^2 - r_1^2)h \\
 &= \pi(r_2 + r_1)(r_2 - r_1)h \\
 &= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1)
 \end{aligned}$$

If we let  $\Delta r = r_2 - r_1$  (the thickness of the shell) and  $r = \frac{r_2 + r_1}{2}$  (the average radius of the shell), then this formula for the volume of a cylindrical shell becomes

$$V = 2\pi r h \Delta r$$

and it can be remembered as

$$V = [\text{circumference}][\text{height}][\text{thickness}]$$

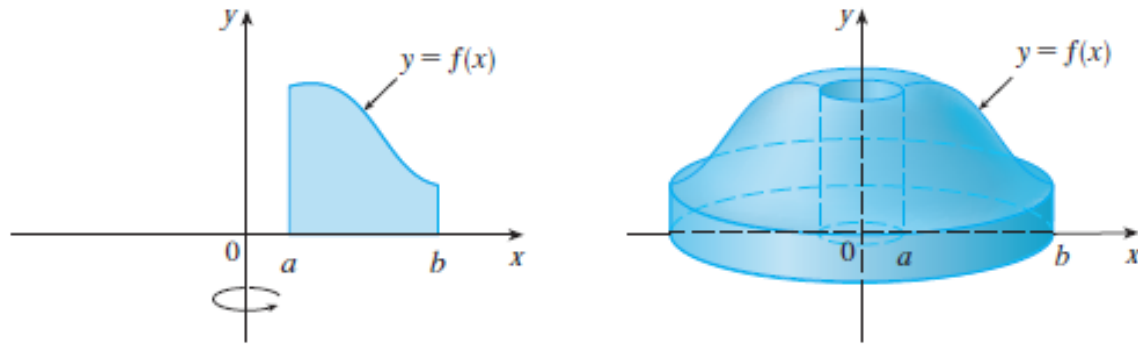


FIGURE 3

Now let  $S$  be the solid obtained by rotating about the  $y$  –  $axis$  the region bounded by  $y = f(x)$  ( $f(x) \geq 0$ ),  $y = 0$ ,  $x = a$ ,  $x = b$ , where  $b > a \geq 0$  (See Figure 3).

We divide the interval  $[a, b]$  into  $n$  subintervals  $[x_{i-1}, x_i]$  of equal width  $\Delta x$  and let  $\bar{x}_i$  be the midpoint of the  $i$ th subinterval. If the rectangle with base  $[x_{i-1}, x_i]$  and height  $f(\bar{x}_i)$  is rotated about the  $y$  –  $axis$ , then the result is a cylindrical shell with average radius  $\bar{x}_i$ , height  $f(\bar{x}_i)$ , and thickness  $\Delta x$  (see Figure 4), so by Formula 1 its volume is

$$V_i = (2\pi\bar{x}_i)[f(\bar{x}_i)]\Delta x$$

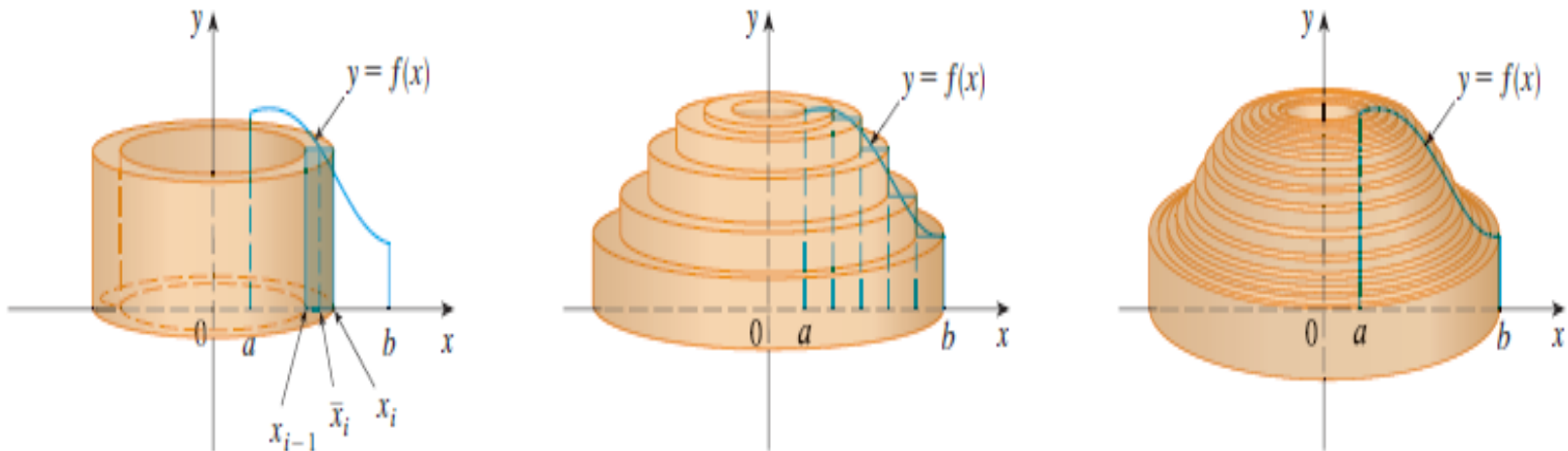


Figure 4

Therefore an approximation to the volume  $V$  of  $S$  is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^n V_i = \sum_{i=1}^n (2\pi\bar{x}_i)[f(\bar{x}_i)]\Delta x$$

This approximation appears to become better as  $n \rightarrow \infty$ . But, from the definition of an integral, we know that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (2\pi\bar{x}_i)[f(\bar{x}_i)]\Delta x = \int_a^b 2\pi x f(x) dx$$

The volume of the solid in Figure 3, obtained by rotating about the  $y$ -axis the region under the curve  $y = f(x)$  from  $a$  to  $b$ , is

$$V = \int_a^b 2\pi x f(x) dx \quad (2)$$

where  $0 \leq a < b$ .

The best way to remember Formula 2 is to think of a typical shell, cut and flattened as in Figure 5, with radius  $x$ , circumference  $2\pi x$ , height  $f(x)$ , and thickness  $\Delta x$  or  $dx$ :

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

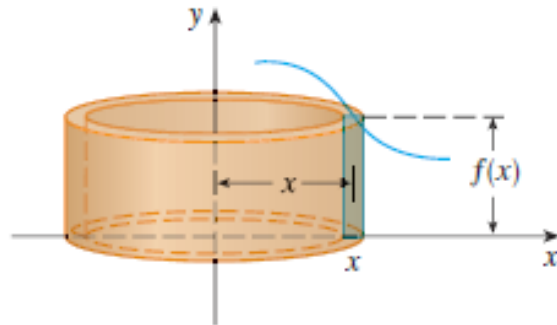
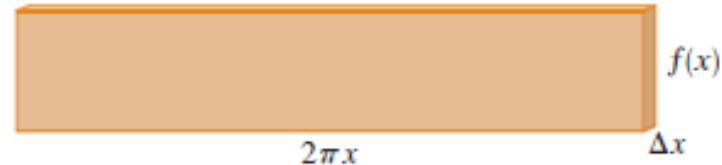


FIGURE 5



Similarly if the region bounded by  $x = f(y)$ ,  $x = 0$  is revolved about the  $x$  - axis,  $c \leq y \leq d$ , then

$$V = 2\pi \int_c^d yf(y)dy$$

## Example -1

Find the volume of the solid obtained by rotating about the  $y$ -axis the region bounded by  $y = 2x^2 - x^3$  and  $y = 0$ .

## Solution

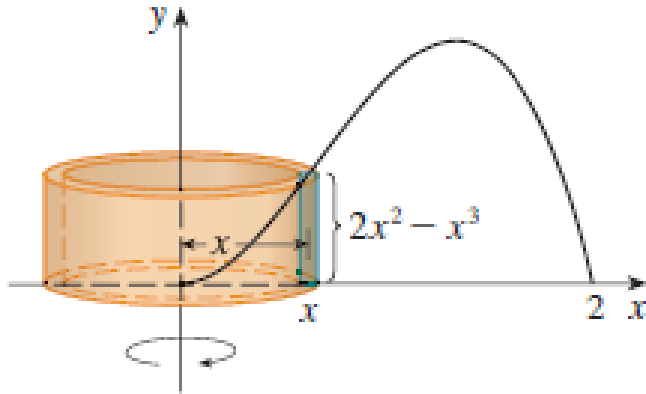


FIGURE 6

**NOTE** Comparing the solution of Example 1 with the remarks at the beginning of this section, we see that the method of cylindrical shells is much easier than the washer method for this problem. We did not have to find the coordinates of the local maximum and we did not have to solve the equation of the curve for  $x$  in terms of  $y$ . However, in other examples the methods of the preceding section may be easier.

## Example -2

Find the volume of the solid obtained by rotating about the  $y$ -axis the region between  $y = x$  and  $y = x^2$ .

## Solution

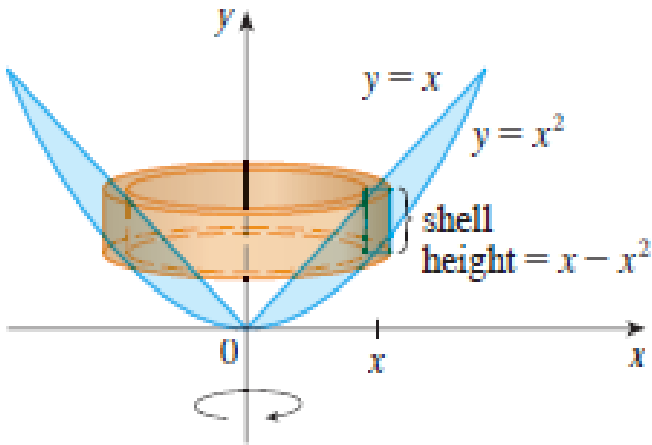


FIGURE 8

As the following example shows, the shell method works just as well if we rotate about the  $x$ -axis. We simply have to draw a diagram to identify the radius and height of a shell.



### Example -3

Use cylindrical shells to find the volume of the solid obtained by rotating about the  $x$ -axis the region under the curve  $y = \sqrt{x}$  from 0 to 1.

### Solution

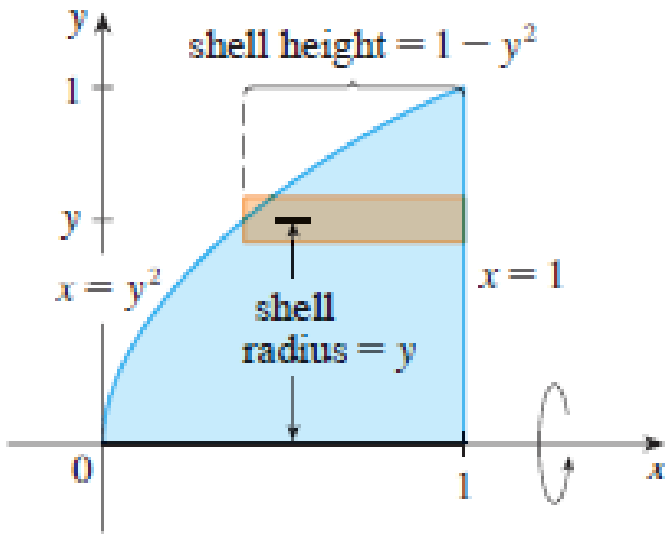


FIGURE 9