2.3 Arc Length

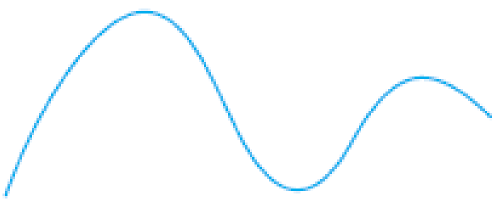

FIGURE 1

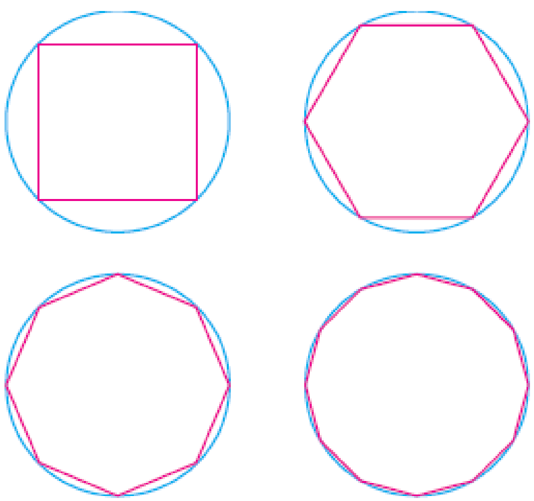

FIGURE 2

What do we mean by the **length of a curve**? We might think of fitting a piece of string to the curve in Figure 1 and then measuring the string against a ruler. But that might be difficult to do with much accuracy if we have a complicated curve.

If the curve is a polygon, we can easily find its length; we just add the lengths of the line segments that form the polygon. (We can use the distance formula to find the distance between the endpoints of each segment.) We are going to define the length of a general curve by first approximating it by a polygon and then taking a limit as the number of segments of the polygon is increased. This process is familiar for the case of a circle, where the circumference is the limit of lengths of inscribed polygons (see Figure 2).

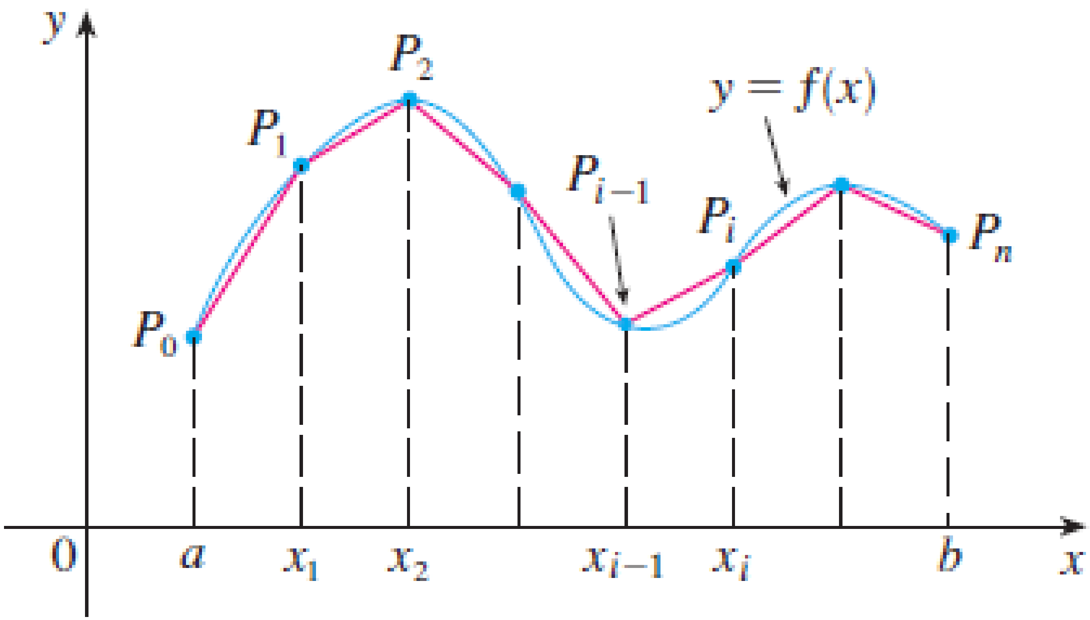

Figure 3

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

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Then there is a number c in (a, b) such that

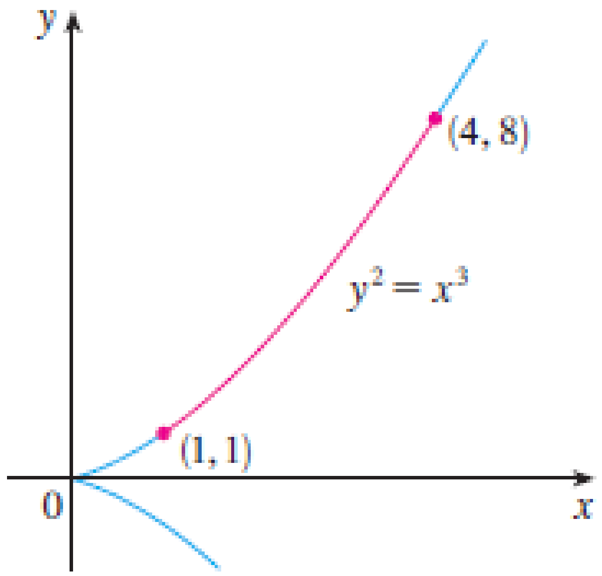
$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

Example 1 Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1,1) and (4,8).

Solution



2.4 Area of a Surface of Revolution

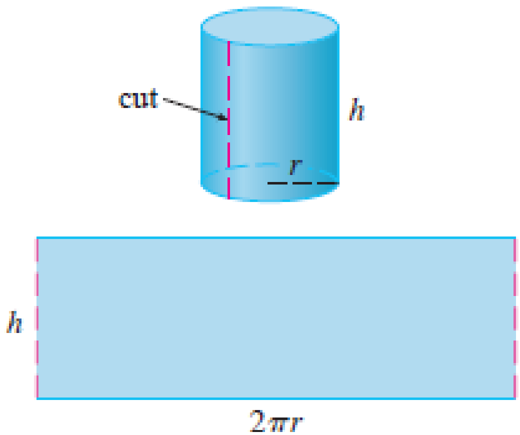


FIGURE 1

A surface of revolution is formed when a curve is rotated about a line. Such as a sphere, a cone, a cylinder …

Let's start with some simple surfaces. The lateral surface area of a circular cylinder with radius r and height h is taken to be $A = 2\pi rh$ because we can imagine cutting the cylinder and unrolling it (as in Figure 1) to obtain a rectangle with dimensions $2\pi r$ and h.

Likewise, we can take a circular cone with base radius r and slant height l, cut it along the dashed line in Figure 2, and flatten it to form a sector of a circle with radius l and central angle $\theta = \frac{2\pi r}{l}$.

We know that, in general, the area of a sector of a circle with radius l and angle θ is $\frac{1}{2}l^2\theta$, and so in this case the area is

$$A = \frac{1}{2}l^2\theta = \frac{1}{2}l^2\left(\frac{2\pi r}{l}\right) = \pi r l$$

Therefore we define the lateral surface area of a cone to be $A = \pi r l$.

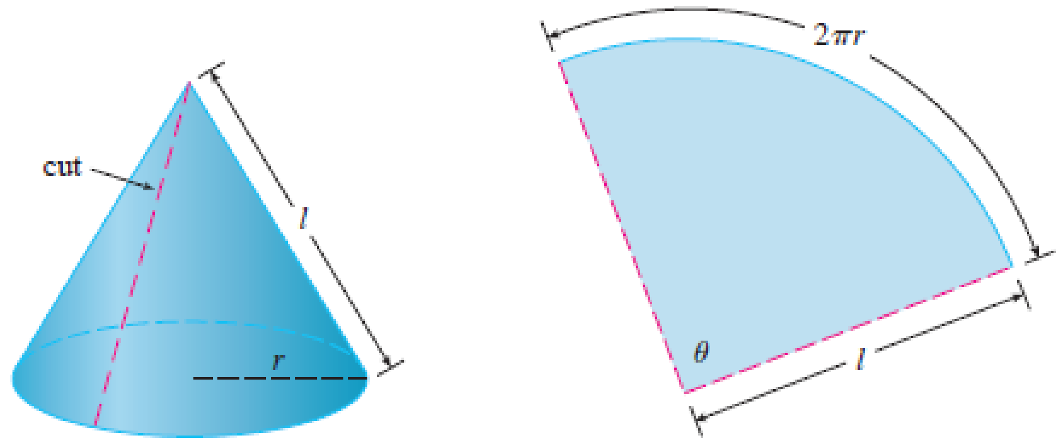


Figure 2

What about more complicated surfaces of revolution? If we follow the strategy we used with arc length, we can approximate the original curve by a polygon. When this polygon is rotated about an axis, it creates a simpler surface whose surface area approximates the actual surface area. By taking a limit, we can determine the exact surface area.

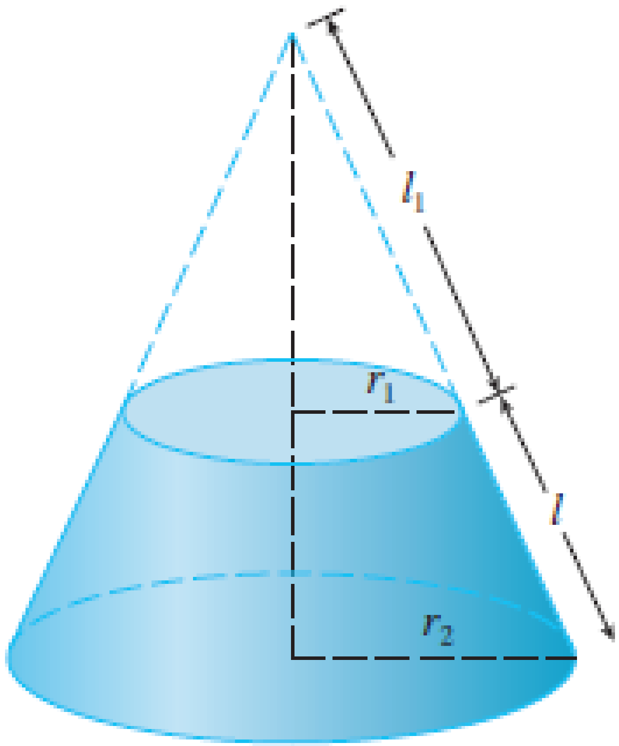
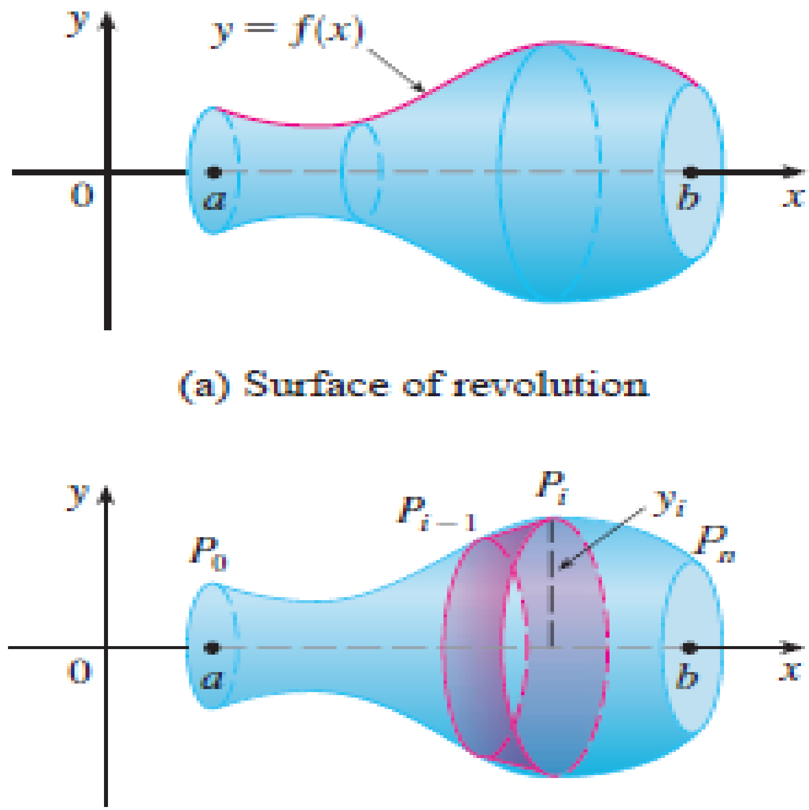


FIGURE 3



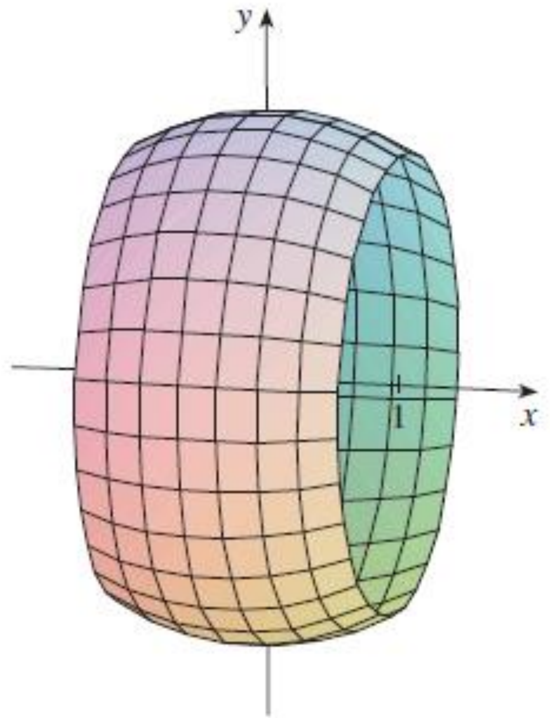
(a) Surface of revolution

(b) Approximating band

FIGURE 4

Example 1

The curve $y=\sqrt{4-x^2}$, $-1\le x\le 1$, is an arc of the circle $x^2+y^2=4$. Find the area of the surface obtained by rotating this arc about the x-axis.



Example 2

The arc of the parabola $y = x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis. Find the area of the resulting surface.

Figure 7 shows the surface of revolution whose area is computed in Example 2.

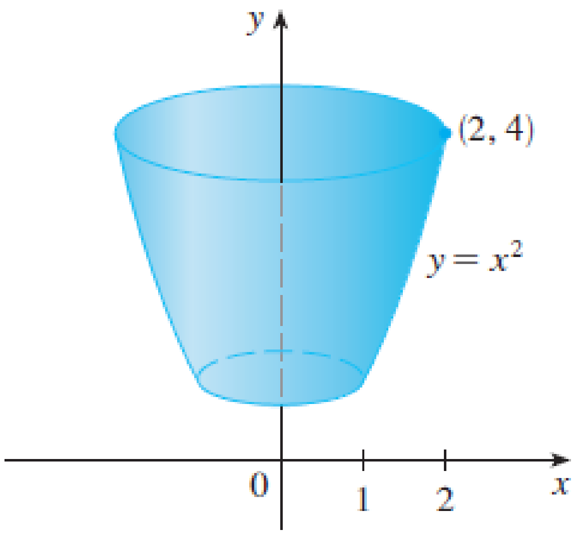


FIGURE 7

2.5 Calculating Some Limits by Using Integral

Theorem

$f:[a,b] \to \mathbb{R}$ *be a continuous function. Then*

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) = \int_a^b f(x)dx.$$