Static Electric Currents

Consider a group of charged particles (each has charge q) of number density N (m^{-3}) moving across an elemental surface $S\hat{n}$ (m^2) with velocity $\vec{v} = v\hat{v}$ (m/sec) in $\hat{v}$ −direction. Within a time interval Δt, the amount of charge ΔQ passing through the surface is equal to the total charge within a differential parallel-piped of volume:

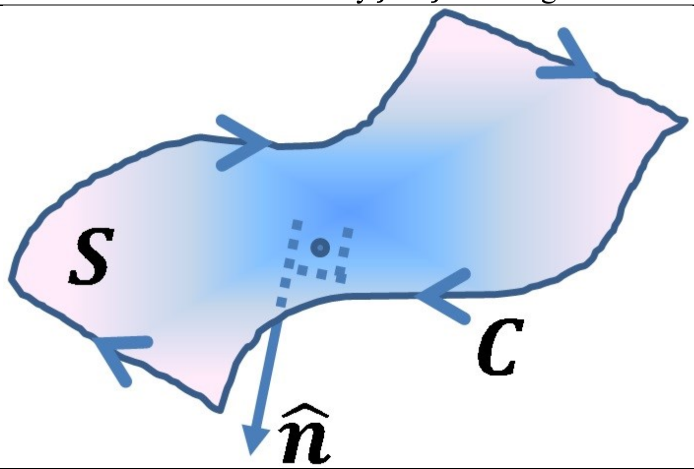
$$\Delta V = \vec{v}.\Delta t \bullet S\hat{n}$$

$$\Delta Q = \Delta V.Nq$$

$$\Delta Q = v.S.\Delta t.Nq$$

$$I = \frac{\Delta Q}{\Delta t} = \frac{v.S.\Delta t.Nq}{\Delta t} = v.S.Nq$$

Interpretting $I\hat{v}$ as the flux of current density $\vec{J} = J\hat{v}$ through surface S



$$I = \int_S \vec{J} \bullet \overrightarrow{dS} = \int_S J\hat{v} \bullet \hat{v} dS = J.S$$

$$J = \frac{I}{S}$$

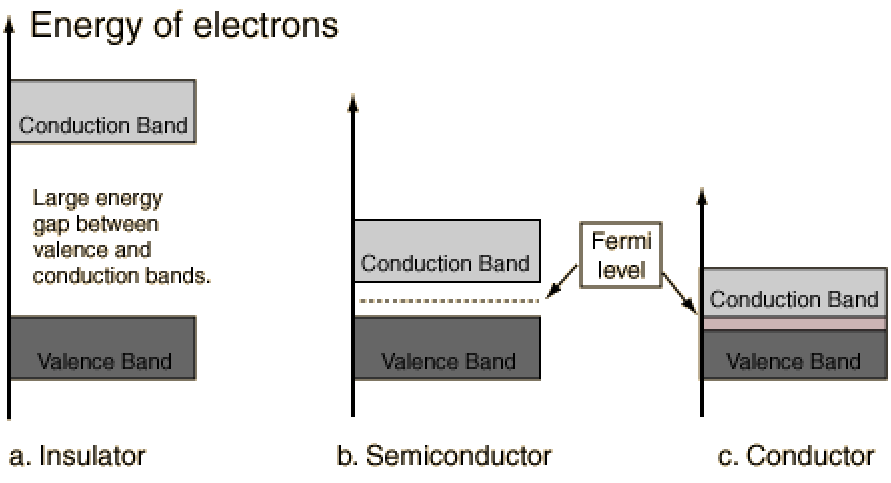
$\vec{J} = J\hat{v} = \frac{I}{S}\hat{v}$			

Convection Currents

Convection currents result from motion of charged particles (e.g. electrons, ions) in “vacuum” (e.g. cathode ray tube), involving with mass transport but without collision. In vacuum-tube diodes, some of the electrons boiled away from the incandescent cathode are attracted to the anode due to theexternal electric field, resulting in a convection current flow. Find the relation between the steady-state current density J and the bias voltage V0 . Assume the electrons leaving the cathode have zero initial velocity. This is the “space-charge limited condition”, arising from the fact that a cloud of electrons (space charges) is formed near the hot cathode, repulsing most of the newly emitted electrons.

Conduction (drift) currents

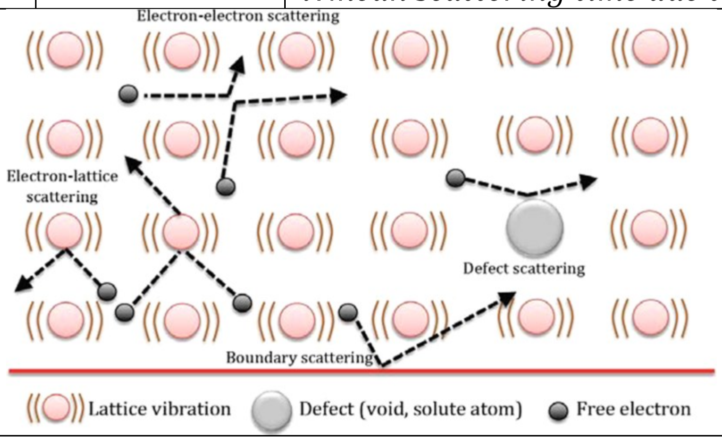
The electrons of conductors only partially fill the conduction band:



These electrons can be easily released from parent nuclei as free electrons by thermal excitation at room temperatures. The velocities of individual free electrons are high in magnitude (~105 m/s at 300K) but random in direction, resulting in no net "drift" motion nor net current.

In the presence of static electric field E , the free electrons experience:

Electric force:	$\vec{F_e} = -q\overrightarrow{E}$	(acceleration)	
Frictional force	$\vec{F_f} = -\frac{m_e \overrightarrow{v}_d}{\tau}$	m_e: *mass of electron* $\overrightarrow{v}_d$: *average drift velocity* τ: *mean scattering time due to crystal lattice*	



In steady state, these two forces balance with each other (Drude model), ⇒

	$\vec{F_e} = -q\overrightarrow{E} = \vec{F_f} = -\frac{m_e \overrightarrow{v}_d}{\tau}$		
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⇒	$\overrightarrow{v}_d = \frac{q\tau}{m_e}\vec{E} = \vec{E}$		
	$\mu_e = \frac{q\tau}{m_e}$	μ_e: *mobility of electrons* $\frac{m^2}{V.sec}$	

μ_e defines describes how easy an external electric field can influence the motion of electrons in the conductor. For typical conductors and strength of electric fields, $|\mu_e| \sim \frac{mm}{sec}$ and is much slower than the speed of individual electrons. The conduction current density is:

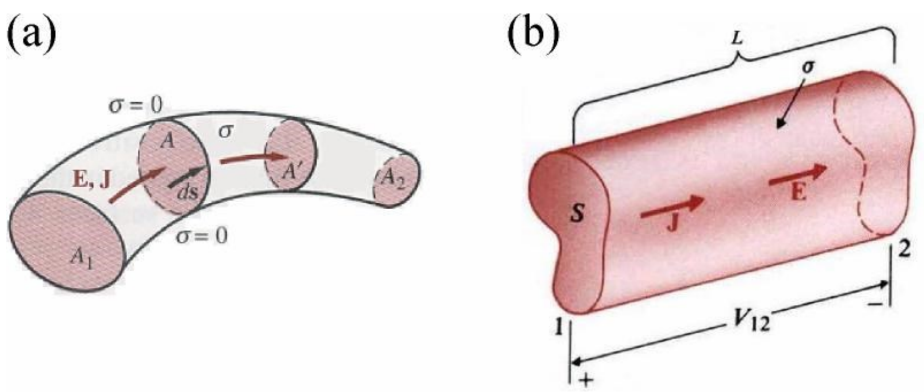
	$\vec{J_e} = \sigma\vec{E}$	$\left(\frac{A}{m^2}\right)$	
conductivity:	$\sigma = \rho_{ve}\mu_e$	$\left(\frac{1}{\Omega m}\right)$	
free electron density:	ρ_{ve}	$\left(\frac{Coul}{m^3}\right)$	
For semiconductors, both electrons and holes contribute to conduction currents,			
⇒	$\sigma = \rho_{ve}\mu_e + \rho_{vh}\mu_h$		

Typical carrier number densities, mobilities, conductivities (below THz)

	μ_e	μ_h	N_e (m^{-3})	N_h (m^{-3})	σ (S/m)
pure Ge	0.39	0.19	2.4×10^{19}	2.4×10^{19}	2.2
pure Si	0.14	0.05	1.4×10^{16}	1.4×10^{16}	4.4×10^{-4}
Cu	0.0032	—	1.13×10^{29}	—	5.8×10^{7}
Al	0.0015	—	1.46×10^{29}	—	3.5×10^{7}
Ag	0.005	—	7.74×10^{28}	—	6.2×10^{7}
Au					4.5×10^{7}

Microscopic and Macroscopic Current Laws

$\vec{J_e} = \sigma\vec{E}$ is the microscopic form of Ohm's law. Consider a piece of (imperfect) conductor of arbitrary shape and homogeneous (finite) conductivity σ:



The potential difference between the two equipotential end faces A1 , A2 is:

$$\Delta V = V(A1) - V(A2) = -\int_{A1}^{A2} \vec{E} \bullet \overrightarrow{dl}$$

where L is some path starting from A1 and ending at A2 . The total current flowing through some surface A between A1 and A2:

$$I = \int_{A} \vec{J} \bullet \overrightarrow{dS}$$

The resistance R of the conductor is defined as:

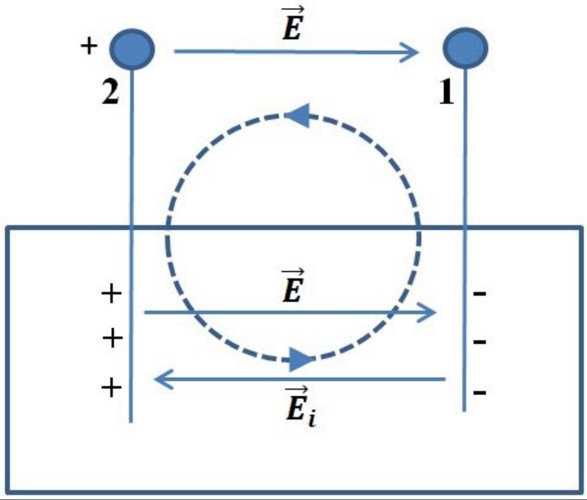
$$R = \frac{\Delta V}{I} = \frac{-\int_{A1}^{A2} \vec{E} \bullet \overrightarrow{dl}}{\int_{A} \vec{J} \bullet \overrightarrow{dS}}$$

which is a constant independent of ΔV and I (but depending on the geometry and material of the conductor). For a conductor of "uniform" cross-sectional area S, assuming conduction current density $\vec{J}$ which is driven by a conservative electric field E (created by charges alone) :

$$\vec{J} = \sigma \vec{E} \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma}$$

$$\oint_{C} \vec{E} \bullet \overrightarrow{dl} = 0 = \oint_{C} \frac{\vec{J}}{\sigma} \bullet \overrightarrow{dl}$$

No steady loop current can exist. Therefore, a non-conservative field produced by batteries, generators …etc. is required to drive charge carriers in a closed loop.



Consider an open-circuited battery, where some positive and negative charges are accumulated in electrodes 1 and 2 due to chemical reaction. Inside the battery, an impressed field E_i (not an electric field, but a “force”) produced by chemical reaction balances the electrostatic field E arising from the accumulated charges, preventing charges from further movement.

$\Delta V = V(2) - V(1) = -\int_{1}^{2} \vec{E} \cdot \vec{dl}$	
$\Delta V = V(2) - V(1) = -\int_{-}^{+} \vec{E} \cdot \vec{dl}$	
$emf = \int_{-}^{+} \vec{E}_i \cdot \vec{dl}$	

The electromotive force (emf), defined as the line integral of E_i from electrode 2 to electrode 1 describes the strength of the non-conservative source:

=> $E= -E_i$ inside the battery.

$\oint_C \vec{E}\bullet\vec{dl} = 0$	$= \left(\int_{inside} \vec{E}\bullet\vec{dl}\right) + \left(\int_{outside} \vec{E}\bullet\vec{dl}\right)$	
	$= \left(\int_{+}^{-} \vec{E}\bullet\vec{dl}\right) + \left(\int_{-}^{+} \vec{E}\bullet\vec{dl}\right)$	
	$= \left(\int_{+}^{-} \vec{E}\bullet\vec{dl}\right) - \Delta V$	
	$= \left(\int_{-}^{+} -\vec{E}\bullet\vec{dl}\right) - \Delta V$	
	$= \left(\int_{-}^{+} \vec{E}_i\bullet\vec{dl}\right) - \Delta V$	
0	$= emf - \Delta V$	
emf	$= \Delta V$	
emf	$= \Delta V = \left(\int_{-}^{+} \vec{E}_i\bullet\vec{dl}\right) = -\int_{-}^{+} \vec{E}\bullet\vec{dl}$	

If the two terminals are connected by a uniform conducting wire of resistance $\boldsymbol{R}$, the total field:

	$\vec{E}_{total} = \begin{cases} \vec{E} + \vec{E}_i = 0 \\ \vec{E} \end{cases}$	$,$ *inside the battery* $,$ *outside the battery*	

drives a loop current *I* of current density $\vec{J} = \frac{I}{S}\hat{l}$ and

$\Delta V = -\int_{1}^{2} \vec{E}\bullet\vec{dl} = -\int_{-}^{+} \vec{E}\bullet\vec{dl} = -\int_{-}^{+} \frac{\vec{J}}{\sigma}\bullet\vec{dl}$	

$$\Delta V = -\int_L^0 \frac{\frac{I}{S}\hat{l}}{\sigma} \cdot \overrightarrow{dl} = +\int_0^L \frac{\frac{I}{S}\hat{l}}{\sigma} \cdot \overrightarrow{dl}$$				
$$\Delta V = +\int_0^L \frac{\frac{I}{S}\hat{l}}{\sigma} \cdot \hat{l}dl = \frac{I}{\sigma S}\int_0^L dl = \frac{I}{\sigma S}\frac{L}{\sigma S}$$				
$$R = \frac{\Delta V}{I} = \frac{L}{\sigma S}$$			(Ω)	

For a closed path with multiple sources and resistors, we get the Kirchhoff's voltage law:

$$\sum_k \Delta V_k = R_k I_k$$	**Kirchhoff's Voltage Law**	

Equation of continuity and Kirchhoff's Current Law:

Consider a net charge Q confined in a volume *V*, bounded by a closed surface *S* . Based on the principle of conservation of charge (a fundamental postulate of physics), a net current flowing out of *V* must result in decrease of the enclosed charge:

$$I = -\frac{dQ}{dt}$$

$$\oint_S \vec{J} \cdot \overrightarrow{dS} = -\frac{d}{dt}\left(\int_V \rho_v dV\right)$$

$$\oint_S \vec{J} \cdot \overrightarrow{dS} = \left(\int_V -\frac{d\rho_v}{dt} dV\right)$$

$\oint_S \vec{J} \cdot \overrightarrow{dS} = \int_V \nabla \cdot \vec{J} dV$	**Divergence Theorem**
$\oint_S \vec{J} \cdot \overrightarrow{dS} = \int_V \nabla \cdot \vec{J} dV = \left(\int_V -\frac{d\rho_v}{dt} dV\right)$	$\forall V$

$$\nabla \cdot \vec{J} = -\frac{d\rho_v}{dt} \Rightarrow$$

$\nabla \cdot \vec{J} + \frac{d\rho_v}{dt} = 0$	**Continuity Equation** **Conservation of charge**

For steady state,

$$\frac{d}{dt} \to 0$$

For steady currents

$$\frac{d\rho_v}{dt} = 0$$

Thus,

$$\nabla \cdot \vec{J} = 0$$

This means there is no steady current source/sink, and the field lines of J always close upon themselves. The total current flowing out of a circuit junction enclosed by surface S becomes:

By Divergence Theorem			
	$\oint_S \vec{J} \bullet \overrightarrow{dS} = \int_V \nabla \bullet \vec{J} dV = \int_V 0 dV = 0$		
	$0 = \oint_S \vec{J} \bullet \overrightarrow{dS} = \sum_k I_k$		
	$\sum_k I_k = 0$	**Kirchhoff's Current Law**	

The Kirchhoff's Current Law is the **macroscopic** form of $\nabla \bullet \vec{J} + \frac{d\rho_v}{dt} = 0$ in steady state.

Example:

Show the dynamics (time dependence) of free charge density ρ inside a homogeneous conductor with constant electric conductivity σ and permittivity ε

Solution:

	$\nabla \bullet \vec{J} = \nabla \bullet \sigma \vec{E}$	
Assuming simple medium	$\nabla \bullet \vec{J} = \sigma \nabla \bullet \vec{E}$	
	$\nabla \bullet \vec{J} = \sigma \nabla \bullet \vec{E} = -\frac{d\rho_v}{dt}$	
	$\nabla \bullet \vec{E} = -\frac{1}{\sigma}\frac{d\rho_v}{dt}$	

	$\nabla \bullet \vec{D} = \rho_v$	
	$\nabla \bullet \varepsilon \vec{E} = \rho_v$	ε_0
Assuming simple medium	$\nabla \bullet \varepsilon \vec{E} = \varepsilon \nabla \bullet \vec{E} = \rho_v$	
	$\nabla \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$	
	$\frac{\rho_v}{\varepsilon} = -\frac{1}{\sigma}\frac{d\rho_v}{dt}$	
	$\frac{d\rho_v}{dt} + \frac{\rho_v}{\frac{\varepsilon}{\sigma}} = 0$	
	$\tau = \frac{\varepsilon}{\sigma}$ **(sec)**	
	$\frac{d\rho_v}{dt} + \frac{\rho_v}{\tau} = 0$	
	$\rho_v = \rho_{v0} e^{-\frac{t}{\tau}}$	

Time constant τ represents the time interval that is needed for ρ_v to drop from ρ_{v0} to $\frac{\rho_{v0}}{e}$ for every point in volume V. For a good conductor like copper, $\tau = 10^{-19}\ sec$.