MAGNETIC SCALAR AND VECTOR POTENTIALS

We recall that some electrostatic field problems were simplified by relating the electric Potential V to the electric field intensity **E** (**E** = $-\nabla V$). Similarly, we can define a potential associated with magnetostatic field B. In fact, the magnetic potential could be scalar V_m vector A. To define V_m and A involves two important identities:

$$\nabla \times (\nabla V) = 0 \tag{1.26}$$

$$\nabla \cdot (\nabla \times A) = 0 \tag{1.27}$$

which must always hold for any scalar field V and vector field A.

Just as **E = -∇V**, we define the *magnetic scalar potential V_m* (in amperes) as related to **H** according to

$$\mathbf{H} = -\nabla \mathbf{V_m} \quad \text{if } \mathbf{J} = 0 \tag{1.28}$$

The condition attached to this equation is important and will be explained. Combining eq. (1.28) and eq. (1.19) gives

$$\mathbf{J} = \nabla \times \mathbf{H} = -\nabla \times (-\nabla V_m) = 0 \tag{1.29}$$

since V_m, must satisfy the condition in eq. (1.26). Thus the magnetic scalar potential V_m is only defined in a region where J = 0 as in eq. (1.28). We should also note that V_m satisfies Laplace's equation just as V does for electrostatic fields; hence,

$$\nabla^2 V_m = 0, \quad (\mathbf{J} = 0) \tag{1.30}$$

SCALAR MAGNETIC POTENTIAL

Like scalar electrostatic potential, it is possible to have scalar magnetic potential. It is defined in such a way that its negative gradient gives the magnetic field, that is,

$$\mathbf{H} = \nabla V_m \tag{3.16}$$

V_m = scalar magnetic potential (Amp)

Taking curl on both sides, we get

$$\nabla \times \mathbf{H} = -\nabla \times \nabla V_m \tag{3.17}$$

But curl of the gradient of any scalar is always zero.

So, $$\nabla \times \mathbf{H} = 0 \tag{3.18}$$

But, by Ampere's circuit law $\nabla \times \mathbf{H} = \mathbf{J}$

or, $$J = 0$$

In other words, scalar magnetic potential exists in a region where $J = 0$.

$$H = -\nabla V_m \quad (J=0) \tag{3.19}$$

The scalar potential satisfies Laplace's equation, that is, we have

$$\nabla.B = \mu_0\ \nabla.H = 0 = m\nabla\ (-\nabla V_m) = 0$$

or,

$$\nabla^2\ V_m = 0\ \mathbf{(J = 0)} \tag{3.20}$$

Characteristics of Scalar Magnetic Potential (V_m)

1. The negative gradient of V_m gives H, or $H = -\nabla V_m$
2. It exists where J = 0
3. It satisfies Laplac's equation.
4. It is directly defined as

$$V_m = -\int_A^B H \cdot dL$$

5. It has the unit of Ampere.

VECTOR MAGNETIC POTENTIAL

Vector magnetic potential exists in regions where J is present. It is defined in such a way that its curl gives the magnetic flux density, that is,

$$\mathbf{B} \equiv \nabla \times \mathbf{A} \tag{3.21}$$

where **A** = vector magnetic potential (wb/m).

It is also defined as

$$A \equiv \oint \frac{\mu_0 IdL}{4\pi R}\left(\frac{Henry - Amp}{m}\right)$$

(3.22)

or, $A \equiv \int_s \frac{\mu_0 Kds}{4\pi R},$ (K = current sheet) (3.23)

or, $A \equiv \oint_v \frac{\mu_0 Jdv}{4\pi R},$ (3.24)

Characteristics of Vector Magnetic Potential

1. It exists even when J is present.
2. It is defined in two ways

   B ≡ ∇ x A and

   $$\int_v \frac{\mu_0 Jdv}{4\pi R}$$

3. $\nabla^2 \mathbf{A} = \mu_0 j$
4. $\nabla^2 \mathbf{A} = 0$ if $J = 0$

5. Vector magnetic potential, **A** has applications to obtain radiation characteristics of antennas, apertures and also to obtain radiation leakage from transmission lines, waveguides and microwave ovens.
6. A is used to find near and far-fields of antennas.

Problem 4:

The vector magnetic potential, A due to a direct current in a conductor in free space is given by A = $(X^2 + Y^2)$ $\underline{\mathbf{a}_Z}$ µwb /m^2. Determine the magnetic field produced by the current element at (1, 2, 3).

Solution:

$$\mathbf{A} = (x^2 + y^2)\, \mathbf{a}_z \ \mu\text{wb/m}^2$$

We have **B** = ∇ x **A**

$$= 10^{-6} \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (x^2 + y^2) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial x}(x^2 + y^2)a_x + \left(-\frac{\partial}{\partial x}\right)(x^2 + y^2)a_y \right] \times 10^{-6}$$

$$= \left[(x^2 + 2y)a_x - (2x + y^2)a_y \right] \times 10^{-6}$$

$$B / at\ (1,2,3) = \left[(1+4)a_x - (2+4)a_y \right] \times 10^{-6}$$

$$= \left(5a_x - 6a_y\right) \times 10^{-6}$$

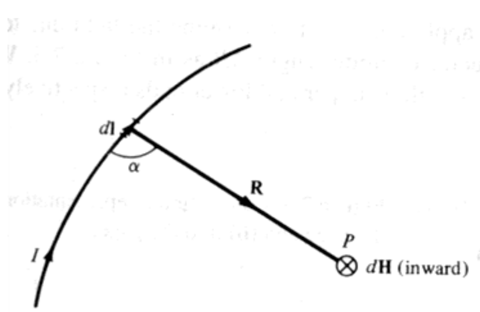
$$H = \frac{1}{\mu_0}\left(5a_x - 6a_y\right) \times 10^{-6}$$

$$= \frac{1}{4\pi \times 10^{-7}}\left(5a_x - 6a_y\right) \times 10^{-6}$$

$\mathbf{H} = (3.978\mathbf{a}_x - 4.774\mathbf{a}_y)$, A/m

BIOT SAVART's LAW

Biot-Savart's law states that the magnetic field intensity ***dH*** produced at a point P, as shown in **the figure**, by the differential current element ***I dl*** is proportional to the product



Magnetic field d**H** at P due to current element *I dl.*

I dl and the sine of the angle $\boldsymbol{\beta}$ between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element. That is,

		$dH \propto \frac{I.dl.sin\beta}{R^2}$	
		$\boldsymbol{dH \propto \frac{I.dl.sin\beta}{R^2}}$	
or		$dH = k\ \frac{I.dl.sin\beta}{R^2}$	*

where, *k* is the constant of proportionality. In SI units, *k* = 1/4π. So, eq. (*) becomes

		$\boldsymbol{dH = \frac{1}{4\pi}\frac{I.dl.sin\beta}{R^2}}$	

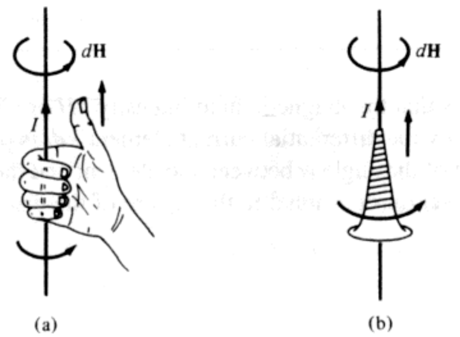
From the definition of cross product equation $\vec{\boldsymbol{A}}$ **x** $\vec{\boldsymbol{B}}$ **= A.B.sinγ.** $\hat{\boldsymbol{n}}$,

		$\boldsymbol{\overrightarrow{dH} = \frac{I.\overrightarrow{dl} \times \hat{R}}{4\pi R^2}}$	

where

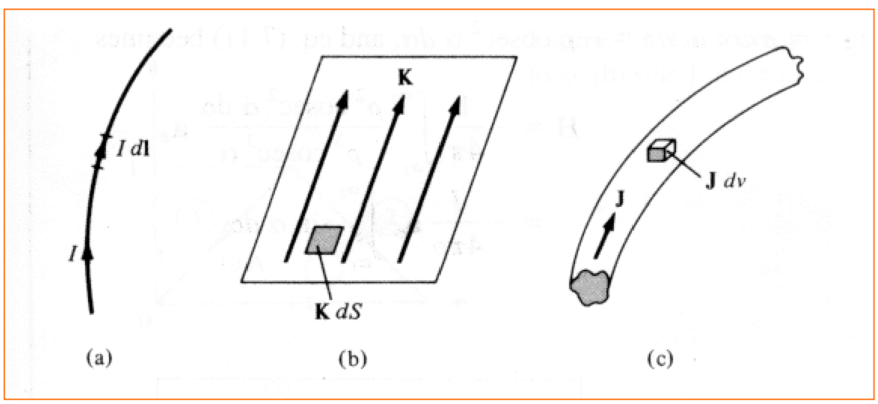
		$\boldsymbol{R = \frac{\vec{R}}{\lvert R \rvert}}$	

Thus, the direction of $\boldsymbol{\overrightarrow{dH}}$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current, the right-hand fingers encircling the wire in the direction of $\boldsymbol{\overrightarrow{dH}}$. Alternatively, one can use the right-handed screw rule to determine the direction of $\boldsymbol{\overrightarrow{dH}}$, with the screw placed along the wire and pointed in the direction of current flow, the direction of advance of the screw is the direction of $\boldsymbol{\overrightarrow{dH}}$



It is customary to represent the direction of the magnetic field intensity $\vec{H}$ (or current I) by a small circle with a dot or cross sign depending on whether $\vec{H}$ (or I) is out of, or into, the page.

As like different charge configurations, one can have different current distributions: line current, surface current and volume current :



If we define $\vec{K}$ as the surface current density (in amperes/meter) and $\vec{J}$ as the volume current density (in amperes/meter square), the source elements are related as

	$I\vec{dl}$	Line Current Density	

$\overrightarrow{dI} =$	$\vec{J}_S dS$	Surface Current Density	
	$\vec{J}_v dV$	Volume Current Density	

Thus, in terms of the distributed current sources, Biot-Savart law becomes:

$$H = \int_L \frac{Idl \times a_R}{4\pi R^2} \qquad \text{(Line current)} \qquad (1.6)$$

$$H = \int_S \frac{KdS \times a_R}{4\pi R^2} \qquad \text{(Surface current)} \qquad (1.7)$$

$$H = \int_V \frac{Jdv \times a_R}{4\pi R^2} \qquad \text{(Volume current)} \qquad (1.8)$$

As an example, let us apply eq. (1.6) to determine the field due to a *straight current* carrying filamentary conductor of finite length AB as in **Figure 1.5**. We assume that the conductor is along the z-axis with its upper and lower ends respectively subtending angles

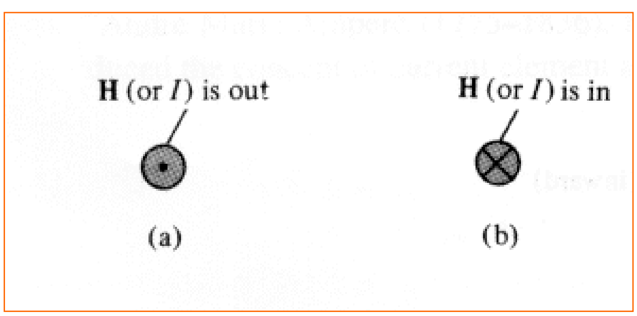


Figure 1.3: Conventional representation of H (or I) **(a)** out of the page and **(b)** into the page.

α_2 *and* α_1 *at* P, the point at which **H** is to be determined. Particular note should be taken of this assumption, as the formula to be derived will have to be applied accordingly. If we consider the contribution d**H** at P due to an element d**l** at (0, 0, z),

$$dH = \frac{Idl \times R}{4\pi R^3} \tag{1.9}$$

But d**l** = dz $\mathbf{a}_z$ and R = $\rho\mathbf{a}_\rho$ - z$\mathbf{a}_z$, so

$$\mathbf{dl} \times \mathbf{R} = \rho \, dz \, \mathbf{a}_\phi \tag{1.10}$$

Hence,

$$H = \int \frac{I\rho \, dz}{4\pi\left[\rho^2 + z^2\right]^{3/2}} a \tag{1.11}$$

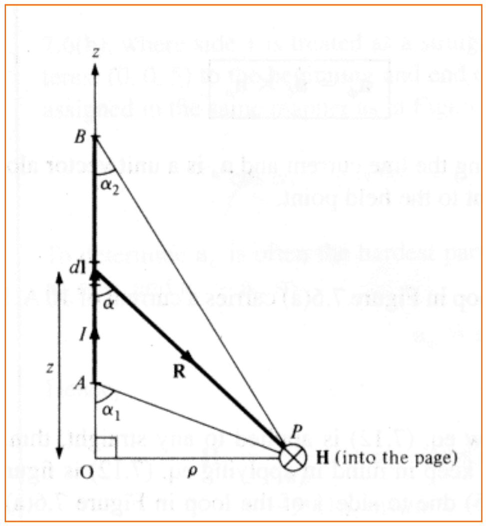


Figure 1.5: Field at point P due to a straight filamentary conductor.

Letting $z = \rho \cot \alpha$, $dz = -\rho \operatorname{cosec}^2 \alpha \, d\alpha$, equation (1.11) becomes

$$H = -\frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \cos ec^2\alpha \ d\alpha}{\rho^3 \cos ec^3\alpha} a_\phi$$

$$= -\frac{I}{4\pi\rho} a_\phi \int_{\alpha_1}^{\alpha_2} \sin \ \alpha \ d\alpha$$

Or

$$H = \frac{I}{4\pi\rho}\left(\cos \ \alpha_2 - \cos \ \alpha_1\right) a_\phi \qquad (1.12)$$

The equation (1.12) is generally applicable for any straight filamentary conductor of finite length. Note from eq. (1.12) that **H** is always along the unit vector $\mathbf{a}_\phi$ (i.e., along concentric circular paths) irrespective of the length of the wire or the point of interest P. As a special case, when the conductor is *semi-infinite* (with respect to P), so that point A is now at O(0, 0, 0) while B is at (0, 0, ∞); $\alpha_1 = 90°$, $\alpha_2 = 0°$, and eq. (1.12) becomes

$$H = \frac{I}{4\pi\rho} a_\phi \qquad (1.13)$$

Another special case is when the conductor is *infinite* in length. For this case, point A is at (0, 0, - ∞) while B is at (0, 0, ∞); $\alpha_1 = 180°$, $\alpha_2 = 0°$. So, eq. (1.12) reduces to

$$H = \frac{I}{2\pi\rho} a_\phi \qquad (1.14)$$

To find unit vector $\mathbf{a}_\phi$ in equations (1.12) to (1.14) is not always easy. A simple approach is to determine $\mathbf{a}_\phi$ from

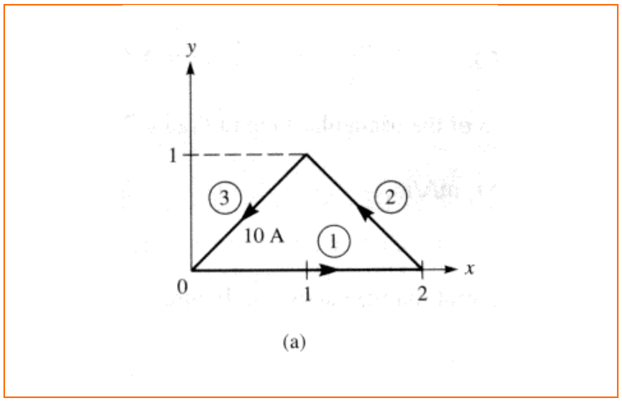
$$a_\phi = a_\ell \times a \qquad (1.15)$$

where $\mathbf{a}_l$ is a unit vector along the line current and $\mathbf{a}_\rho$ is a unit vector along the perpendicular line from the line current to the field point.

Illustration: The conducting triangular loop in Figure 1.6(a) carries a current of 10 A. Find **H** at (0, 0, 5) due to side 1 of the loop.

Solution:

This example illustrates how eq. (1.12) is applied to any straight, thin, current-carrying conductor. The key point to be kept in mind in applying eq. (1.12) is figuring out α_1, α_2, ρ and $\mathbf{a}_\phi$. To find **H** at (0, 0, 5) due to side 1 of the loop in Figure 1.6(a), consider Figure



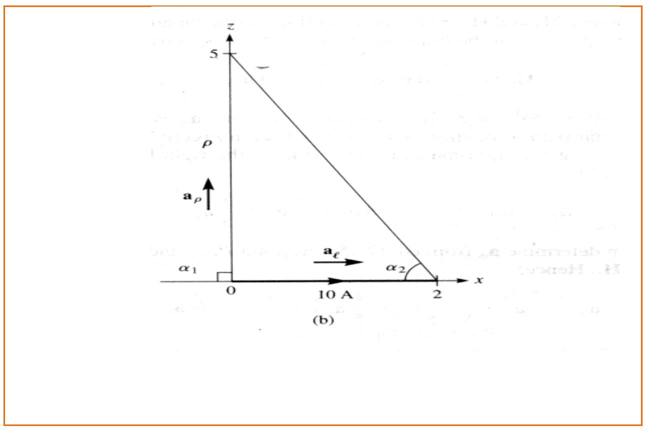


Figure 1.6: (a) conducting triangular loop **(b)** side 1 of the **loop.**

1.6(b), where side 1 is treated as a straight conductor. Notice that we join the Point of interest (0, 0, 5) to the beginning and end of the line current. Observe that α_1, α_2 and ρ are assigned in the same manner as in Figure 1.5 on which eq. (1.12) is based.

$$\cos \alpha_1 = \cos 90^\circ = 0, \quad \cos\alpha_2 = \frac{2}{\sqrt{29}}, \qquad \rho = 5$$

To determine $\mathbf{a}_\phi$ is often the hardest part of applying eq. (1.12). According to eq. (1.15), $\mathbf{a}_l = \mathbf{a}_x$ and $\mathbf{a}_\rho = \mathbf{a}_z$, so

$$\mathbf{a}_\phi = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

Hence,

$$H_1 = \frac{1}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right) a_\phi = \frac{10}{4\pi(5)}\left(\frac{2}{\sqrt{29}} - 0\right)(-a_y)$$

$= -59.1\ \mathbf{a}_y$ mA/m