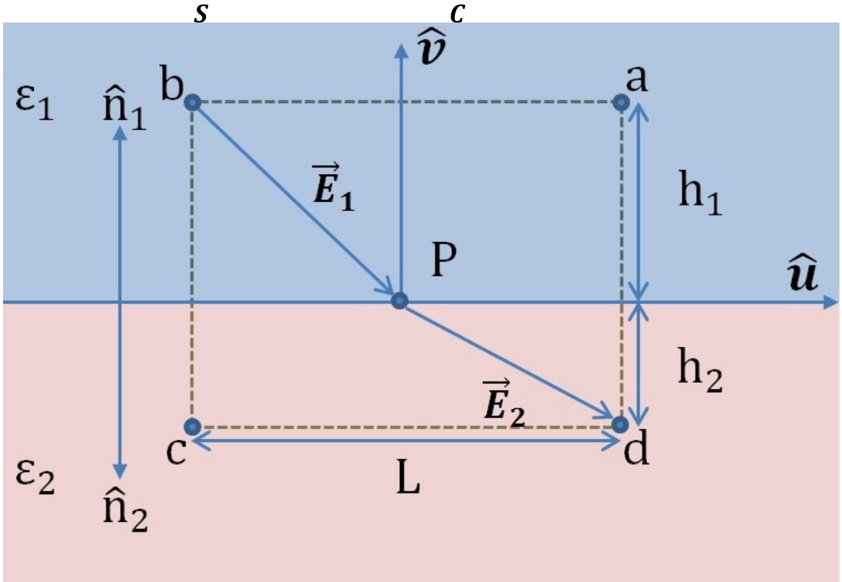
**Boundary Conditions for electrostatic fields**

$$\nabla \times \vec{E} = \vec{0}$$

$$\int_S \nabla \times \vec{E} \bullet \overrightarrow{dS} = \oint_C \vec{E} \bullet \overrightarrow{dl} = 0$$




$$\oint_C \vec{E} \bullet \overrightarrow{dl} = \sum_{i=1}^{4} \int_{C_i} \vec{E} \bullet \overrightarrow{dl}$$

$$\oint_C \vec{E} \bullet \overrightarrow{dl} = \int_a^b \vec{E} \bullet \overrightarrow{dl} + \int_b^c \vec{E} \bullet \overrightarrow{dl} + \int_c^d \vec{E} \bullet \overrightarrow{dl} + \int_d^a \vec{E} \bullet \overrightarrow{dl} +$$

$$\int_b^c \vec{E} \bullet \overrightarrow{dl} = \int_d^a \vec{E} \bullet \overrightarrow{dl} = 0$$

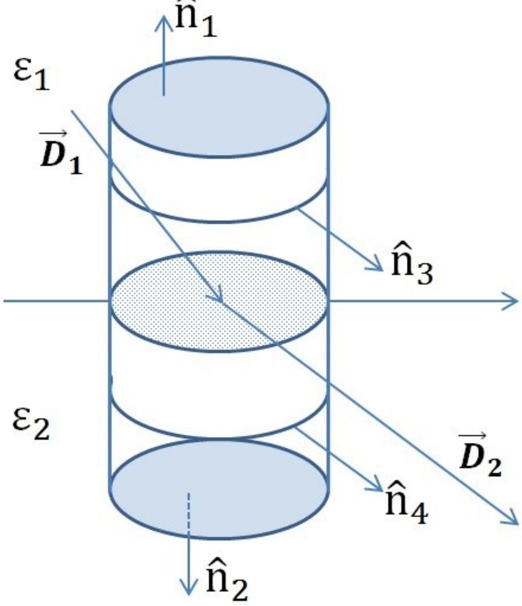
$$\int_a^b \vec{E}_1 \bullet \overrightarrow{dl} + \int_c^d \vec{E}_2 \bullet \overrightarrow{dl} = 0$$

$$E_{1//}(-L) + E_{2//}(-L) = 0$$

$$E_{1//} = E_{2//}$$

Tangential component of $\vec{E}_1$ is equal to tangential component of $\vec{E}_2$

**Boundary Condition 2**




$$\oint_S \vec{E} \cdot \overrightarrow{dS} = \int_V \frac{\rho_v}{\varepsilon_0} dV$$

$$\oint_S \vec{E} \cdot \overrightarrow{dS} = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\oint_S \vec{D} \cdot \overrightarrow{dS} = Q_{enclosed} = \rho_S S$$

$$\oint_S \vec{D} \cdot \overrightarrow{dS} = \int_{S_1} \vec{D}_1 \cdot \overrightarrow{dS} + \int_{S_2} \vec{D}_2 \cdot \overrightarrow{dS} + \int_{S_3} \vec{D}_1 \cdot \overrightarrow{dS} + \int_{S_4} \vec{D}_2 \cdot \overrightarrow{dS}$$

$$\int_{S_3} \vec{D}_1 \cdot \overrightarrow{dS} \to 0$$

$$\int_{S_4} \vec{D}_2 \cdot \overrightarrow{dS} \to 0$$

$$\oint_S \vec{D} \cdot \overrightarrow{dS} = \rho_S S = \int_{S_1} \vec{D}_1 \cdot \overrightarrow{dS} + \int_{S_3} \vec{D}_3 \cdot \overrightarrow{dS}$$

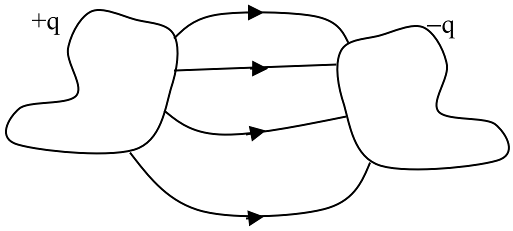
$$D_{1\perp}(-S) + D_{2\perp}(+S) = \rho_S S$$

$$=> D_{2\perp} - D_{1\perp} = \rho_S$$

$$=> \hat{n}_1 \cdot \left(\vec{D}_2 - \vec{D}_1\right) = \rho_S$$

$$=> \hat{n}_2 \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_S$$

# CAPACITANCE




A set of conductors can store electric charge. The net charge $Q$=0=$q$–$q$, but the magnitude of charge on each conductor is $|q|$.

This charge $q$ is proportional to the potential difference between the conductors:

$$q = C\Delta V \qquad \Delta V = V_+ - V_- \equiv V$$
$$q = CV$$

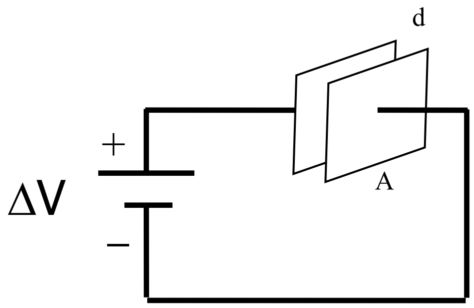
The constant of proportionality between charge and potential difference is $C$≡capacitance. Unit is Farad (F) ≡ Coulomb/Volt.

$$1\mu\text{F} = 10^{-6}\ \text{F}$$
$$1\text{pF} = 10^{-12}\ \text{F}$$

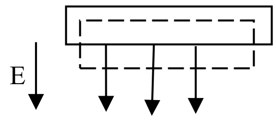
To set up a potential difference between 2 conductors requires an electric “pump”, such as a battery.

A larger capacitance implies that a large charge $q$ is stored for the same potential difference $V$. Capacitance depends only on the geometry of the conductors, not the charge $q$ or voltage $V$. We can see this through examples.

## Parallel Plate Capacitor




Consider the top view of the 2 plates:




Create a Gaussian surface (box) that extends inside and outside one of the conductor surfaces.

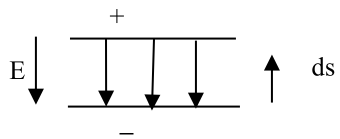
Gauss' Law $\Rightarrow \Phi = \oint \qquad \frac{q}{\varepsilon_0}$

$\mathbf{E} = 0$ inside a conductor
$\mathbf{E} \cdot d\mathbf{A} = 0$ on left/right edges
$|\mathbf{E}| \neq 0$ on front outside face only

$$\Rightarrow \Phi = \oint \qquad = \frac{q_{\text{enc}}}{\varepsilon_0}$$

$$\Rightarrow q = \varepsilon_0 E A$$

The electric potential difference between the 2 plates is given by:




$\Delta V = V_+ - V_- = -\int_-^+ \mathbf{E} \cdot d\mathbf{s} \qquad \mathbf{E} \cdot d\mathbf{s} = -|\mathbf{E}|\, ds$ opposite directions

$$\Delta V = |\mathbf{E}|\, d$$

$$\Rightarrow |\mathbf{E}| = \frac{\Delta V}{d}$$

So for parallel plates:

$$q = \varepsilon_0 E A = \varepsilon_0 \frac{\Delta V}{d} A = \left( \varepsilon_0 \frac{A}{d} \right) \Delta V$$

$$q = C \Delta V$$

$$\Rightarrow C = \varepsilon_0 \frac{A}{d} \qquad \varepsilon_0 = 8.85 \times 10^{-12} \ \frac{\text{C}^2}{\text{Nm}^2} = 8.85 \text{ pF/m}$$

## Cylindrical Capacitor (Cable)

Let inner conductor have radius *a*, and outer radius *b*. Take Gaussian surface as cylinder between conductors (**E**=0 inside conductors).

$$\Phi = \square \qquad \frac{q}{\varepsilon_0}$$

$$\Rightarrow E2\pi rL\varepsilon_0 = q \qquad \mathbf{E}\cdot d\mathbf{A} = 0 \text{ on cylinder ends}$$

$$\Rightarrow E = \frac{q}{2\pi\varepsilon_0 L}\frac{1}{r}$$

$$\Delta V = V_+ - V_- = -\int_-^+ \mathbf{E}\cdot d\mathbf{s} \qquad \mathbf{E}\cdot d\mathbf{s} = -|\mathbf{E}|\,ds \quad \text{opposite directions, but } ds = -dr \text{ opposite again}$$

$$\Delta V = -\int_-^+ |\mathbf{E}|\,dr = \int_a^b |\mathbf{E}|\,dr = \int_a^b \frac{q}{2\pi\varepsilon_0 L}\frac{dr}{r}$$

$$\Delta V = \frac{q}{2\pi\varepsilon_0 L}\ln\frac{b}{a}$$

$$\Rightarrow q = \left(\frac{2\pi\varepsilon_0 L}{\ln b/a}\right)\Delta V$$

$$\Rightarrow C = \frac{2\pi\varepsilon_0 L}{\ln b/a}$$

## Spherical Capacitor

Let inner sphere have radius *a*, and outer radius *b*. Take Gaussian surface as sphere between conductors (**E**=0 inside conductors).

$$\text{Gauss' Law} \Rightarrow |\mathbf{E}| = K\frac{q}{r^2} \qquad a < r < b$$

$$\Delta V = V_+ - V_- = -\int_-^+ \mathbf{E}\cdot d\mathbf{s} \qquad \mathbf{E}\cdot d\mathbf{s} = -|\mathbf{E}|\,ds \quad \text{opposite directions, but } ds = -dr \text{ opposite again}$$

$$\Delta V = -\int_-^+ |\mathbf{E}|\,dr = \int_a^b |\mathbf{E}|\,dr = \int_a^b Kq\frac{dr}{r^2}$$

$$\Delta V = Kq\left(-\frac{1}{r}\right)\Bigg|_a^b = Kq\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Rightarrow q = \left(\frac{1}{K}\frac{ab}{b-a}\right)\Delta V$$

$$\Rightarrow C = \frac{1}{K}\frac{ab}{b-a}$$

## Capacitors in Parallel

Consider $N$ capacitors all connected in parallel to the same source of potential difference $V$. Across each capacitor $i$ the charge on one of the plates is: $q_i = C_i V$

The total charge on all the plates with the same electric potential is:

$$Q = \sum_{i=1}^{N} q_i = \sum_{i=1}^{N} C_i V = V \sum_{i=1}^{N} C_i$$

So we can write the equivalent capacitance $C_{\text{equiv}}$ as:

$$Q = C_{\text{equiv}} V$$
$$C_{\text{equiv}} = \sum_{i=1}^{N} C_i$$

In other words, the equivalent capacitance of N capacitors in parallel is the sum of the individual capacitances. Considering the example of parallel plate capacitors, adding several in parallel is equivalent to extending the area of the plates. Since the capacitance is proportional to the area, it increases in direct proportion.

## Capacitors in Series

For $N$ capacitors in series, the magnitude of the charge $q$ on each plate must be the same. Consider the electric conductor connecting any 2 capacitors, and suppose that a charge +q is on the plate of one of the capacitors the conductor is connected to. Since the conductor was originally uncharged, a charge –q must exist on the plate of the second capacitor. Now a capacitor has the same charge magnitude on each plate, so by inference we can determine that the magnitude of charge on each plate in the series of capacitor must be the same. The potential difference across any capacitor is given by $V_i = \frac{q}{C_i}$. The total potential difference must add up to electric potential supplied by the battery or power supply:

$$V = \sum_{i=1}^{N} \frac{q}{C_i} = \frac{q}{C_{\text{equiv}}}$$

So the equivalent capacitance of capacitors connected in series is given by:

$$\frac{1}{C_{\text{equiv}}} = \sum_{i=1}^{N} \frac{1}{C_i}$$

The potential difference across any capacitor can be determined by:

$$q = \frac{V}{\sum_{j=1}^{N} \frac{1}{C_j}}$$

$$V_i = \frac{q}{C_i}$$

**Energy Stored in a Capacitor**

Let's calculate the work required of a battery or power supply to move an infinitesimal charge $dq'$ onto the plate of a capacitor already containing a charge $q'$. This is the same as finding the change in the potential energy of the capacitor. Recall that the electric potential difference across a device is equal to the potential energy difference per unit charge:

$$\Delta V = \frac{\Delta U}{q}$$

The potential energy difference is equal to the negative of the work done by the electric field to set up the configuration, or in other words equal to the work done by the power supply or battery to move the charge (the charge must move against the direction of the electric field):

$$W_{\text{app}} = \Delta U = q\Delta V$$

So the work done to move an infinitesimal charge $dq'$ onto the plate of a capacitor is given by:

$$dW_{\text{app}} = dq'\Delta V$$

If the capacitor already has a charge $q'$, then $\Delta V = \dfrac{q'}{C}$

$$\text{So } dW_{\text{app}} = \frac{q'}{C}dq'$$

So to charge up a capacitor initially uncharged to a total charge $q$ will require integrating over the above expression:

$$W_{\text{app}} = \int dW_{\text{app}} = \frac{1}{C}\int_0^q q'dq' = \frac{1}{C}\frac{q^2}{2}$$

$$W_{\text{app}} = \Delta U = \frac{q^2}{2C}$$

Since $q = C\Delta V$ for a capacitor, the electric potential energy stored in a capacitor can be expressed in 2 ways:

$$\Delta U = \frac{q^2}{2C} = \frac{1}{2}C\left(\Delta V\right)^2$$

This potential energy can be used to perform work if the capacitor is disconnected from the power supply and connected to an electrical circuit. For example, a flash bulb on a camera works in this way. Using both forms of the relation for the energy in a capacitor, we can see which capacitor has a greater energy when two are connected in series or parallel. When two capacitors are in series, each has the same charge $q$ on one of the plates. Thus by $\Delta U = \dfrac{q^2}{2C}$, the smaller capacitance has the greater energy stored. For two capacitors in parallel, both capacitors have the same voltage across the plates. Thus by $\Delta U = \dfrac{1}{2}C\left(\Delta V\right)^2$, the larger capacitance stores the greater energy.

## Energy Stored in an Electric Field

Let's apply the expression for the potential energy to the specific example of a parallel plate capacitor with plate area *A* and plate separation *V*. The capacitance is given by:

$$C = \varepsilon_0 \frac{A}{d}$$

The magnitude of the electric field between the plates is given by $|E| = \frac{V}{d}$. So the potential energy stored in the capacitor is

$$U = \frac{1}{2} C (\Delta V)^2 = \frac{\varepsilon_0 A |E|^2 d^2}{2d}$$

and per unit volume *V=Ad*, The energy density is given by

$$u = \frac{1}{2} \varepsilon_0 |E|^2$$

This result is more general—it applies to any capacitor. Even more, one can interpret the result as saying the potential energy of the capacitor is stored in the electric field of the capacitor. The electric field has a reality to it, and contains an energy density given by the above expression. The field is able to do work on electric charges by expending this potential energy.

## Boundary Value Problems

We will spend some time in looking at the mathematical foundations of electrostatics. For a charge distribution defined by a charge density □□, the electric field in the region is given by $\nabla \cdot \vec{E} = \rho/\epsilon_0$, which gives, for the potential $\varphi$, the equation which is known as the **Poisson's equation**,

$$\nabla^2 \varphi = -\frac{\rho}{\epsilon_0}$$

In particular, in a region of space where there are no sources, we have

$$\nabla^2 \varphi = 0$$

Which is called the **Laplace's equation**. In addition, under static conditions, the equation

$$\nabla \times \vec{E} = 0$$

is valid everywhere. Solutions of Laplace's equation are known as **Harmonic functions**. The expressions for the Laplacian operator in Cartesian, spherical and cylindrical coordinates are given by the following expressions

(It may be noted that we have used the symbol $\varphi$ for the potential and a closely similar symbol $\phi$ for the azimuthal angle of the spherical coordinate system, which should not cause confusion).

The formal solution of Poisson's equation has been known to us from our derivation of the form of potential using Coulomb's law. It is easy to check that the expression

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{All\ Space} \frac{\rho(\overrightarrow{r'})}{\left|\vec{r} - \overrightarrow{r'}\right|} d^3r'$$

Where we have used the primed variable $\overrightarrow{r'}$ to indicate the integrated variable and $\vec{r}$ as the position coordinate of the point where the potential is calculated. We will operate both sides of the above equation with $\nabla^2$. Since $\nabla^2$ acts on the variable $\overrightarrow{r'}$, we can take it inside the integral on the right and make it operate on the function $\frac{\mathbf{1}}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}'}|}$ , and get,

$$\nabla^2 \varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{All\ Space} \rho(\overrightarrow{r'}) \nabla^2 \left(\frac{1}{\left|\vec{r} - \overrightarrow{r'}\right|}\right) d^3r'$$

Recall that $\nabla^2\left(\frac{1}{|\vec{r}-\vec{r'}|}\right) = -4\pi\delta^3(\vec{r} - \overrightarrow{r'})$, so that the integral on the right is easily computed using the property of the delta function to be $-\frac{\rho}{\epsilon_0}$.

**Uniqueness Theorem :**

There may be many solutions to Poisson's and Laplace's equations. However, the solutions that interest us in Physics are those which satisfy the given boundary conditions. Other than the assumed existence of the solutions subject to given boundary conditions, one theorem that comes to our help is what is known as the Uniqueness Theorem. The theorem basically states that corresponding to various possible solutions of Laplace's or Poission's equation, the solution that satisfies the given boundary condition is unique, i.e., no two different solutions can satisfy the given equation with specified boundary conditions.

The great advantage of the theorem lies in the fact that if we obtain a solution by some technique or even by intuition, we need not look any further, the solution that we have at hand is the only possible solution.

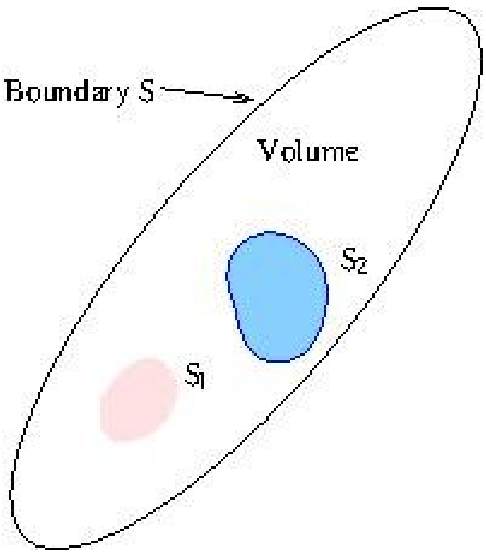
What are these boundary conditions? Consider, for instance, a situation in which we are required to find the solution of either of these equations in a region of volume V bounded by a surface S. Within this region, we have, for instance, a set of conductors with surfaces $S_1$, $S_2$, … on which we know the values of the potential. In addition, we are given the potential function on the surface S itself. This is an example of boundary condition, known as the **Dirichlet boundary condition.**

It is also possible that instead of specifying the values of potential on the conductors, we are given the electric fields on the surface of the conductors, which as we know, are directed normal to the surface of the conductors. This would be yet another example of boundary conditions. Boundary conditions where the normal derivatives of the function to be derived are specified on some surface or surfaces are known as the **Neumann boundary condition**.

In addition to these two types of boundary conditions, there are other possible boundary conditions. For instance in **Cauchy boundary condition**, the value of both the potential and its normal derivative are specified. There could be **mixed boundary conditions** in which different types of boundary conditions may be specified on different parts of the boundary.

Uniqueness theorem applies to those cases where there is only one type of boundary condition, viz. either Dirichlet or Neumann boundary condition.

To prove the uniqueness theorem, let us assume that contrary to the assertion made in the theorem, there exist two solutions $\varphi_1$and$\varphi_2$ of either Poisson's or Laplace's equation which satisfy the same set of boundary conditions on surfaces $S_1$, $S_2$, … and the boundary S. The conditions, as stated above, may be either of Dirichlet type or Neumann type :




$$\varphi_1|_{S_1} = \varphi_2|_{S_1}, \varphi_1|_{S_2} = \varphi_2|_2, \ldots \varphi_1|_S = \varphi_2|_S$$

OR

$$\left.\frac{\partial\varphi_1}{\partial n}\right|_{S_1} = \left.\frac{\partial\varphi_2}{\partial n}\right|_{S_1}, \left.\frac{\partial\varphi_1}{\partial n}\right|_{S_2} = \left.\frac{\partial\varphi_2}{\partial n}\right|_{S_2}, \ldots, \left.\frac{\partial\varphi_1}{\partial n}\right|_{S} = \left.\frac{\partial\varphi_2}{\partial n}\right|_{S}$$

Let us define a new function $\varphi = \varphi_1 - \varphi_2$. In view of the fact that $\varphi_1$and$\varphi_2$ satisfy the same boundary conditions, the boundary condition satisfied by $\varphi$ are $\varphi|_{S_1} = \varphi|_{S_2} = \cdots = \varphi|_S = 0$ (Dirichlet) OR $\left.\frac{\partial\varphi}{\partial n}\right|_{S_1} = \left.\frac{\partial\varphi}{\partial n}\right|_{S_2} = \ldots\left.\frac{\partial\varphi}{\partial n}\right|_{S} = 0$ (Neumann).

Further, whether $\varphi_1$and$\varphi_2$ satisfy Poisson's or Laplace's equation, their difference $\varphi = \varphi_1 - \varphi_2$ satisfies Laplace's equation.

In order to prove the Uniqueness theorem we will use Green's First Identity, derived in Module 1, which states that for two arbitrary scalar fields $\phi$ and $\psi$, the following identity holds,

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi) d^3r = \oint_S \phi \frac{\partial\psi}{\partial n} dS$$

where S is the boundary defining the volume V. We choose $\phi = \psi = \varphi$, to get,

$$\int_V (\varphi\nabla^2\varphi + |\nabla\varphi|^2) d^3r = \oint_S \varphi \frac{\partial\varphi}{\partial n} dS$$

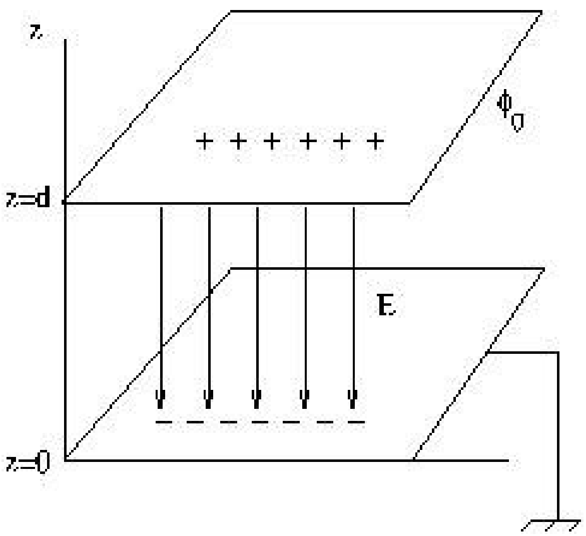
Since on the surface , either the Dirichlet or the Neumann boundary condition is valid, the right hand side of the above is zero everywhere. (Note that the surface consists of $S + S_1 + S_2 + \cdots$, with the direction of normal being outward on S and inward on the conductor surfaces enclosed).

Since $\varphi$ satisfies Laplace's equation, we are then left with $\int_V |\nabla\varphi|^2 d^3r = 0$. The integrand, being a square of a field is positive everywhere in the volume and its integral can be zero only if the integral itself is identically zero. Thus we have, $\nabla\varphi = 0$ , which leads to$\varphi =$ constant everywhere on the volume. If Dirichlet boundary condition is satisfied, then $\varphi = 0$ on the surface and therefore, it is zero everywhere in the volume giving, $\varphi_1 = \varphi_2$. If, on the other hand, Neumann boundary condition is satisfied, we must have $\frac{\partial\varphi}{\partial n} = 0$, i.e. $\varphi = \varphi_1 - \varphi_2 =$ constant everywhere. Since the constant can be chosen arbitrarily, we take the constant to be zero and get $\varphi_1 = \varphi_2$.

**Example 1** : Parallel Plate Capacitor :

Consider a parallel plate capacitor with a plate separation of d between the plates. The lower plate is grounded while the upper plate is maintained at a constant potential $\varphi_0$. Between the two plates there are no sources and hence Laplace's equation is valid in this region. Since the plates are assumed infinite in the x, y directions, the only variation is with respect to the z direction and we have,

$$\nabla^2\varphi \equiv \frac{\partial^2\varphi}{\partial z^2} = 0$$




The solution of this equation is straightforward, and we get,

$$\varphi(z) = Az + B$$

where A and B are constants. Substituting the boundary conditions at z=0 and at z=d,

$$\varphi(z = 0) = 0 = B$$

$$\varphi(z = d) = \varphi_0 = Ad$$

which gives, $A = \frac{\varphi_0}{d},\ B = 0$

Substituting these, we get

$$\varphi(z) = \frac{\varphi_0}{d}z$$

The electric field in the region is given by $\vec{E} = -\nabla\varphi = -\hat{k}\frac{\partial\varphi}{\partial z} = -\frac{\varphi_0}{d}\hat{k}$. This shows that the electric field between the plates is constant and is directed from the upper plate toward the lower plate. The upper plate gets positively charged and the lower plate is negatively charged. We can find out the charge densities on the plates by taking the normal component of the electric field. For instance on the upper plate, the direction of normal being in the negative z direction, $\hat{n} = -\hat{k}$, we get,

$$\sigma = \epsilon_0 E_n = +\frac{\varphi_0}{d}\epsilon_0$$

Likewise, the normal to the lower plate being in the positive z direction, the charge density on that plate is equal and opposite. If we multiply the charge density by the area of the plate A, we get (area is taken to be large so that the edge effects are neglected),

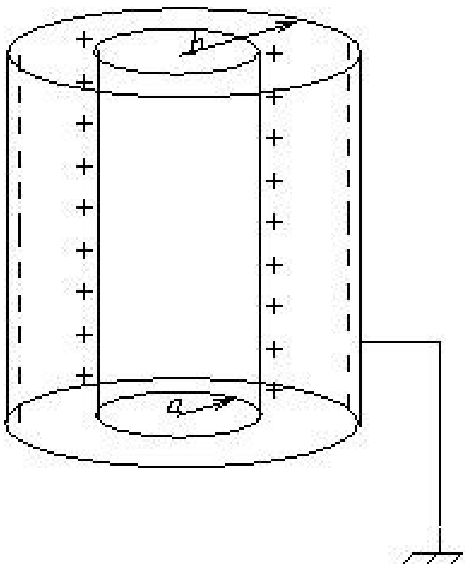
$$Q = A\sigma = A\frac{\varphi_0}{d}\epsilon_0$$

to be the charge on the upper plate. The lower plate has equal but opposite charge. Dividing the amount of charge on the positive plate by the potential difference $\varphi_0$ between the plates, we get the capacitance of the parallel plate capacitor to be $C = \frac{\epsilon_0 A}{d}$.

**Example 2** : Coaxial Cable :

Consider a coaxial cable of inner radius a and outer radius b. The outer conductor is grounded while the inner conductor is maintained at a constant potential $\varphi_0$. Taking the z axis of the cylindrical coordinate system along the axis of the cylinders, we can write down the Laplace's equation in the space between the cylinders as follows :

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$




Here, $\rho$ is the radial distance from the axis and $\theta$ is the polar angle. We assume the cylinders to be of infinite extend in the z direction. This implies that there is azimuthal symmetry and no variation with respect to z as well. The solution can only depend on the radial distance $\rho$. Giving,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) = 0$$

The solution of the above equation is easy to obtain, and we get,

$$\varphi(\rho) = A\ln\rho + B$$

where A and B are two constants which must be determined from the boundary conditions.

$\varphi(\rho = b) = 0$ gives $A\ln b + B = 0$, so that $B = -A\ln b$. The boundary condition at $\rho = b$ gives $\varphi_0 = A\ln a - A\ln b$, so that

$$A = \frac{\varphi_0}{\ln(\frac{a}{b})}$$

Substituting these in the expression for the potential, we get the potential in the region between the cylinders to be given by

$$\varphi(\rho) = \varphi_0\frac{\ln(\frac{\rho}{b})}{\ln\left(\frac{a}{b}\right)}$$

The electric field in the region is given by the negative gradient of the potential which is simply the derivative with respect to $\rho$,

$$\vec{E} = -\nabla\varphi = -\hat{\rho}\frac{\partial}{\partial\rho}\varphi(\rho)$$

$$= -\hat{\rho}\frac{\varphi_0}{\ln(\frac{a}{b})}\frac{1}{\rho}$$

Since $\ln\left(\frac{a}{b}\right) < 0$, the direction of the electric field is outward from the inner conductor, which gets positively charged. The charge density of the inner conductor is given by

$$\sigma = \epsilon_0 E_n = +\varphi_0\epsilon_0\frac{1}{a\ln(\frac{b}{a})}$$

where we have taken care of the minus sign by inverting the argument of log.

Note that the expression for the charge density on the outer plate will not be identical because of the fact that the radii of the two cylinders are different. The total charge per unit length on the inner conductor is given by

$$Q = \varphi_0 \epsilon_0 \frac{2\pi a}{a \ln(\frac{b}{a})} = \frac{2\pi\varphi_0\epsilon_0}{\ln(\frac{b}{a})}$$

The magnitude of the charge per unit length on the outer conductor can be seen to be the same. The capacitance per unit length is thus given by

$$C = \frac{2\pi\epsilon_0}{\ln(\frac{b}{a})}$$

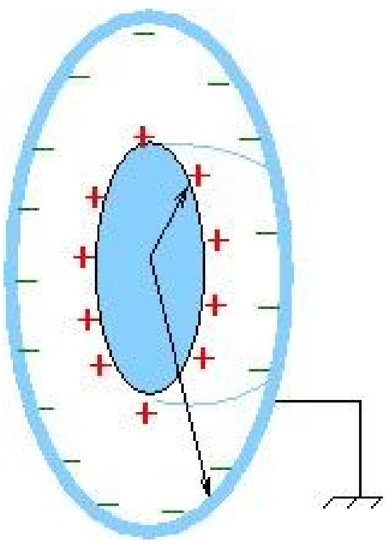
**Example 3** : Spherical Capacitor:

Consider a spherical capacitor with the inner conductor having a radius $a$ and the outer conductor a radius $b$. The outer conductor is grounded and the inner conductor is maintained at a constant potential $\varphi_0$. Because of spherical symmetry, the potential can only depend on the radial distance r. The radial part of Laplace's equation, which is valid in the space between the conductors is given by

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = 0$$

This is a differential equation in a single variable, and the solution can be easily obtained as

$$\varphi(r) = -\frac{A}{r} + B$$



where A and B are constants.

Inserting the boundary conditions,

$$\varphi(r = b) = 0$$

gives $B = \frac{A}{b}$. The other boundary condition $\varphi(r = a) = \varphi_0$ gives

$$\varphi_0 = -\frac{A}{a} + \frac{A}{b}$$

$$A = -\frac{\varphi_0 ab}{b - a}$$

Substituting these into the solution, we have,

$$\varphi(r) = \frac{\varphi_0 ab}{b - a}\left(\frac{1}{r} - \frac{1}{b}\right)$$

As before, we find the charge density on the spheres by taking the normal component of the electric field. The electric field is given by,

$$\vec{E} = -\frac{d\varphi}{dr}\hat{r}$$

$$= \frac{\varphi_0 ab}{b - a}\frac{1}{r^2}\hat{r}$$

The charge density on the inner plate is given by

$$\sigma_{in} = \epsilon_0 \frac{\varphi_0 ab}{b - a}\frac{1}{a^2}$$

The total charge on the inner sphere is

$$Q = \frac{4\pi\epsilon_0\varphi_0 ab}{b - a}$$

It can be checked that the outer sphere has equal and opposite negative charge. The capacitance is given by

$$C = \frac{4\pi\epsilon_0 ab}{b - a}$$

The capacitance of a single spherical conductor is obtained by taking the outer sphere to infinity, i.e., $b \to \infty$, which gives the capacitance for a single conductor to be $4\pi\epsilon_0 a$. This