Joule's law

In the presence of an electric field $\vec{E}$, free electrons in a conductor have a drift (average) velocity, $\vec{v}$. Collisions among free electrons and immobile atoms transfer energy from the electric field to thermal vibration. Quantitatively, the work done by $\vec{E}$ in moving an amount of charge *q in unit volume* for a differential "drift" displacement $\vec{dl} = \vec{v}dt$ is

	$\Delta W = \vec{F} \bullet \vec{dl} = q\vec{E} \bullet \vec{v}\Delta t$		
	$\Delta W = q\vec{E} \bullet \vec{dl} = q\vec{E} \bullet \vec{v}\Delta t$		
Power dissipated is,			
$p =$	$\frac{\Delta W}{\Delta t} = \frac{q\vec{E} \bullet \vec{v}\Delta t}{\Delta t}$		
$p =$	$\frac{q\vec{E} \bullet \vec{v}\Delta t}{\Delta t}$		
$p =$	$q\vec{E} \bullet \vec{v}$		
$p =$	$q\vec{E} \bullet \frac{\vec{J}}{q}$		
$p =$	$\vec{E} \bullet \vec{J}$	**Per unit volume**	
(Ohmic) Power Density			
$p =$	$\vec{E} \bullet \vec{J}$	**(Watts/m^3)**	
Total Power Dissipation with inhomogeneous $\vec{E}(R)$ & $\sigma(R), \vec{J}(R)$			

$$P = \int pdV$$	

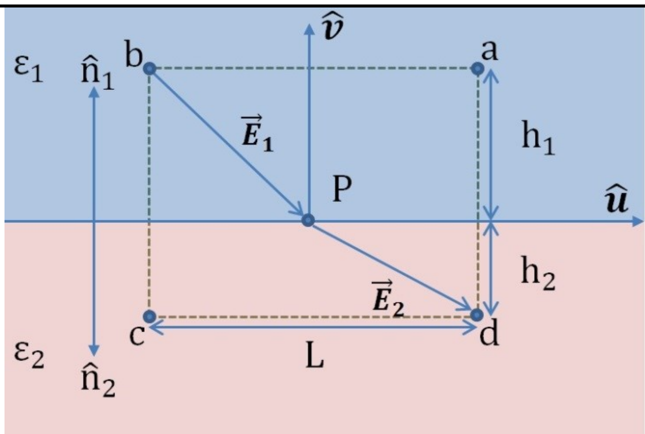
$$P = \int_V \vec{E}(R) \bullet \vec{J}(R)dV$$	**(Watts)**	

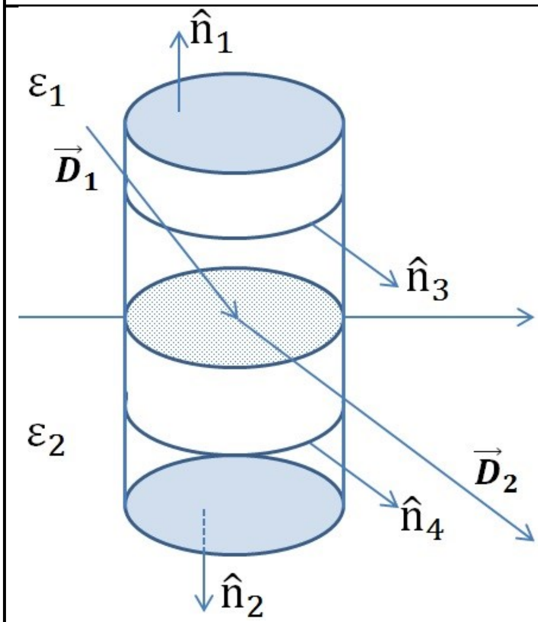
If we apply a voltage difference V_{12} across a homogeneous conductor of uniform cross-sectional area S, conductivity σ and length L,⇒

$$P = \int_V \vec{E}(R) \bullet \vec{J}(R)dV = \int_C \vec{E}(R) \bullet \vec{J}(R)Sdl =$$	
$$P = \int_C \vec{E}(R) \bullet J(R)\hat{l}.S.dl$$	
$$P = \int_C \vec{E}(R) \bullet I\hat{l}dl = \int_C E(R).I.dl =$$	
$$= \int_C \vec{E}(R) \bullet I\hat{l}dl = I\int_C E(R)dl = IV_{12}$$	

Boundary Conditions

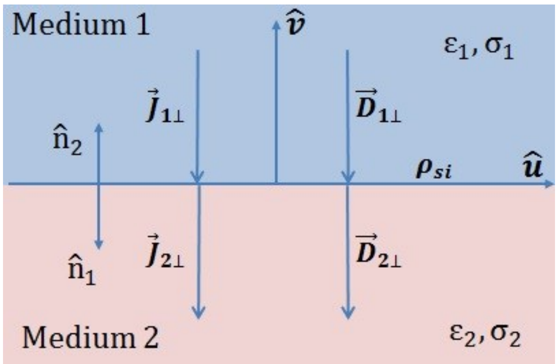
$$\nabla \times \vec{E} = \vec{0}$$	
$$\int_S \nabla \times \vec{E} \bullet \vec{dS} = \oint_C \vec{E} \bullet \vec{dl} = 0$$	

		
$\boldsymbol{E_{1//} = E_{2//}}$		
Tangential component of $\vec{\boldsymbol{E}}_{\boldsymbol{1}}$ is equal to tangential component of $\vec{\boldsymbol{E}}_{\boldsymbol{2}}$		
=>	$\frac{\boldsymbol{J_{1//}}}{\sigma_1} = \frac{\boldsymbol{J_{2//}}}{\sigma_2}$	

Boundary Condition 2		
$\boldsymbol{\nabla \cdot \vec{J} = 0}$		
$\int_V \boldsymbol{\nabla \cdot \vec{J} dV} = \int_V \boldsymbol{0 dV}$		
	$\oint_S \boldsymbol{\vec{J} \cdot \vec{dS}} = \int_V \boldsymbol{0 dV}$ $\oint_S \boldsymbol{\vec{J} \cdot \vec{dS} = 0}$	

$\oint_S \vec{J} \cdot \overrightarrow{dS} = \int_{S_1} \vec{J}_1 \cdot \overrightarrow{dS} + \int_{S_2} \vec{J}_2 \cdot \overrightarrow{dS} + \int_{S_3} \vec{J}_1 \cdot \overrightarrow{dS} + \int_{S_4} \vec{J}_2 \cdot \overrightarrow{dS}$	
$\int_{S_3} \vec{J}_1 \cdot \overrightarrow{dS} \to 0$	
$\int_{S_4} \vec{J}_2 \cdot \overrightarrow{dS} \to 0$	
$\oint_S \vec{J} \cdot \overrightarrow{dS} = 0 = \int_{S_1} \vec{J}_1 \cdot \overrightarrow{dS} + \int_{S_3} \vec{J}_3 \cdot \overrightarrow{dS}$	
$J_{1\perp}(-S) + J_{2\perp}(+S) = 0$	
$=> J_{2\perp} - J_{1\perp} = 0$	
$=> \hat{n}_1 \cdot (\vec{J}_2 - \vec{J}_1) = 0$	
$=> \hat{n}_2 \cdot (\vec{J}_1 - \vec{J}_2) = 0$	

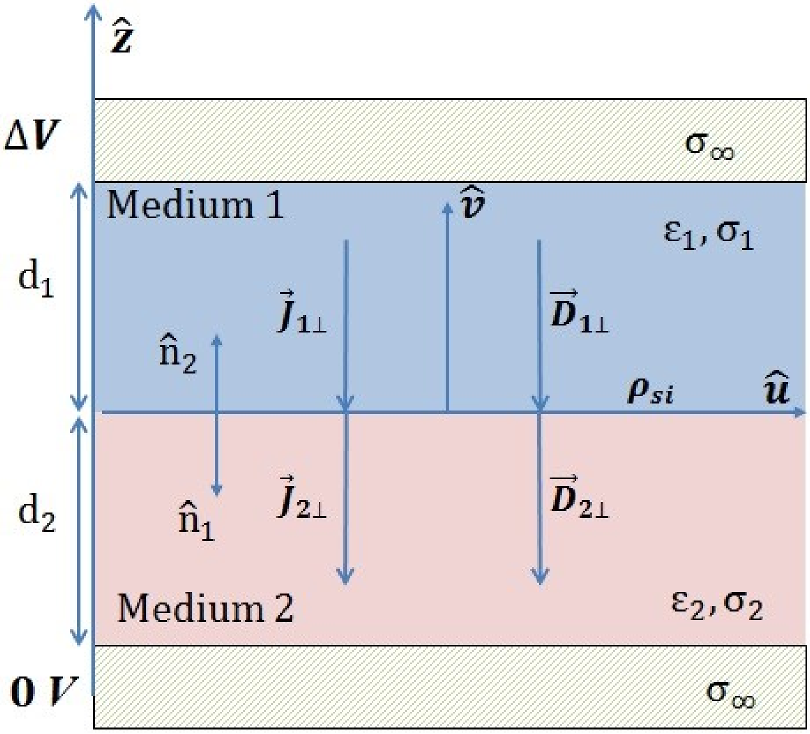
Ex: Find ρ_{si} in terms of $D_{1\perp}, D_{2\perp}, \varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2$



$\vec{D}_{1\perp} = D_{1\perp}(-\hat{v})$	
$\vec{D}_{2\perp} = D_{2\perp}(-\hat{v})$	
$D_{2\perp} - D_{1\perp} = \rho_{Si}$	
$\varepsilon_2 E_{2\perp} - \varepsilon_1 E_{1\perp} = \rho_{Si}$	
$J_{2\perp} - J_{1\perp} = 0$	
$\sigma_2 E_{2\perp} - \sigma_1 E_{1\perp} = 0$	
$E_{1\perp} = \frac{\sigma_2}{\sigma_1} E_{2\perp}$	
$D_{1\perp} = \varepsilon_1 E_{1\perp} = \varepsilon_1 \frac{\sigma_2}{\sigma_1} E_{2\perp}$	
$D_{1\perp} = \frac{\varepsilon_1}{\varepsilon_2}\frac{\sigma_2}{\sigma_1} D_{2\perp}$	
$E_{2\perp} = \frac{\sigma_1}{\sigma_2} E_{1\perp}$	
$D_{2\perp} = \varepsilon_2 E_{2\perp} = \varepsilon_2 \frac{\sigma_1}{\sigma_2} E_{1\perp}$	
$D_{2\perp} = \frac{\varepsilon_2}{\varepsilon_1}\frac{\sigma_1}{\sigma_2} D_{1\perp}$	
$\rho_{Si} = (1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{\sigma_2}{\sigma_1}) D_{2\perp}$	
$\rho_{Si} = (\frac{\varepsilon_2}{\varepsilon_1}\frac{\sigma_1}{\sigma_2} D_{1\perp} - 1) D_{1\perp}$	
$\rho_{Si} = 0$ if $\frac{\varepsilon_2}{\varepsilon_1}\frac{\sigma_1}{\sigma_2} = 1$, $(\frac{\sigma_1}{\varepsilon_1} = \frac{\sigma_2}{\varepsilon_2})$	
Or,	
$\rho_{Si} = 0$ if $\sigma_1 = \sigma_2 = 0$	

Both media is lossless, thus no free charge can exist

Ex: Find J , E in the two lossy media between two parallel conducting plates biased by a dc voltage V_0. Also find the surface charge densities on the two conducting plates and on the interface between the two lossy media.



$$\vec{J}_1 = J_{1\perp}(-\hat{z})$$

$$\vec{J}_1 = J_{2\perp}(-\hat{z})$$

$$J_{1\perp} = J_{2\perp} = J$$

$$\vec{E}_1 = E_1(-\hat{z}) = E_{1\perp}(-\hat{z})$$

$$\vec{E}_1 = E_1(-\hat{z}) = \frac{J_{1\perp}}{\sigma_1}(-\hat{z}) = \frac{J}{\sigma_1}(-\hat{z})$$

$$\vec{E}_2 = E_2(-\hat{z}) = E_{2\perp}(-\hat{z})$$

$\vec{E}_2 = \frac{J_{2\perp}}{\sigma_2}(-\hat{z}) = \frac{J}{\sigma_2}(-\hat{z})$	
$\Delta V - 0 = -\int_{-}^{+} \vec{E} \bullet \vec{dl}$	
$\Delta V = \left(-\int_{0}^{d_2} \vec{E}_2 \bullet \vec{dl}_2\right) + \left(-\int_{d_2}^{d_2+d_1} \vec{E}_1 \bullet \vec{dl}_1\right)$	
$\Delta V = \left(-\int_{0}^{d_2} \vec{E}_2 \bullet dz\hat{z}\right) + \left(-\int_{d_2}^{d_2+d_1} \vec{E}_1 \bullet dz\hat{z}\right)$	
$\Delta V = \left(-\int_{0}^{d_2} \frac{J}{\sigma_2}(-\hat{z}) \bullet dz\hat{z}\right) + \left(-\int_{d_2}^{d_2+d_1} \frac{J}{\sigma_1}(-\hat{z}) \bullet dz\hat{z}\right)$	
$\Delta V = \left(\frac{J}{\sigma_2}\int_{0}^{d_2} (+\hat{z}) \bullet dz\hat{z}\right) + \left(\frac{J}{\sigma_1}\int_{d_2}^{d_2+d_1} (+\hat{z}) \bullet dz\hat{z}\right)$	
$\Delta V = \left(\frac{J}{\sigma_2}\int_{0}^{d_2} dz\right) + \left(\frac{J}{\sigma_1}\int_{d_2}^{d_2+d_1} dz\right)$	
$\Delta V = \left(\frac{J}{\sigma_2} d_2\right) + \left(\frac{J}{\sigma_1} d_1\right)$	
$J = \frac{\Delta V}{\left(\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2}\right)} = \frac{\sigma_1\sigma_2\Delta V}{(\sigma_2 d_1 + \sigma_1 d_2)}$	
$E_1 = \frac{J}{\sigma_1} = \frac{\sigma_2\Delta V}{(\sigma_2 d_1 + \sigma_1 d_2)}$	

$E_2 = \frac{J}{\sigma_2} = \frac{\sigma_1 \Delta V}{(\sigma_2 d_1 + \sigma_1 d_2)}$	
$D_1 = \varepsilon_1 E_1 = \frac{\varepsilon_1 \sigma_2 \Delta V}{(\sigma_2 d_1 + \sigma_1 d_2)}$	
$D_2 = \varepsilon_2 E_2 = \frac{\varepsilon_2 \sigma_1 \Delta V}{(\sigma_2 d_1 + \sigma_1 d_2)}$	
$\hat{n}_{\sigma 1} \bullet (\vec{D}_{\varepsilon_1} - \vec{D}_{\sigma 2}) = \rho_{s1}$	
$-\hat{z} \bullet (\vec{D}_1 - 0) = -\hat{z} \bullet (D_1(-\hat{z}) - 0) = \rho_{s1}$	
$D_1 = \rho_{s1}$	
$\rho_{s1} = D_1 = \frac{\varepsilon_1 \sigma_2 \Delta V}{(\sigma_2 d_1 + \sigma_1 d_2)}$	
$\hat{n}_{\sigma 2} \bullet (\vec{D}_{\varepsilon_2} - \vec{D}_{\sigma 2}) = \rho_{s2}$	
$+\hat{z} \bullet (\vec{D}_2 - 0) = +\hat{z} \bullet (D_2(-\hat{z}) - 0) = \rho_{s2}$	
$D_2 = -\rho_{s2}$	
$\rho_{s2} = -D_2 = -\frac{\varepsilon_2 \sigma_1 \Delta V}{(\sigma_2 d_1 + \sigma_1 d_2)}$	
$\rho_{si} = (1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{\sigma_2}{\sigma_1}) D_{2\perp}$	
$\rho_{si} = (1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{\sigma_2}{\sigma_1}) \frac{\varepsilon_2 \sigma_1 \Delta V}{(\sigma_2 d_1 + \sigma_1 d_2)}$	
$\rho_{si} = \frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2) \Delta V}{(\sigma_2 d_1 + \sigma_1 d_2)}$	

$\lvert\rho_{s1}\rvert \neq \lvert\rho_{s2}\rvert$ but $\rho_{s1} + \rho_{s2} + \rho_{si} = 0$	

Evaluation of Resistance

Resistance of single imperfect conductor

The resistance $\boldsymbol{R}$ of a piece of homogeneous lossy medium of finite conductivity σ can be evaluated by the following steps:

(1) Assume a potential difference $\Delta \boldsymbol{V}$ between the two conductors.

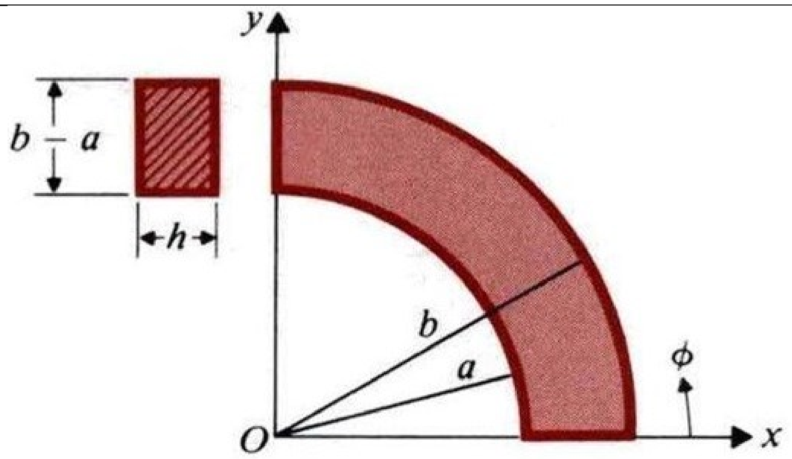
(2). Find the potential distribution $\boldsymbol{\Phi}$ by solving boundary-value problem.

(3) Find $\vec{\boldsymbol{E}}$ by using $\vec{\boldsymbol{E}} = \boldsymbol{\nabla\Phi}$

(4) Find the total current by $\boldsymbol{I} = \int \vec{\boldsymbol{J}} \cdot \overrightarrow{\boldsymbol{dS}} = \int \sigma \vec{\mathrm{E}} \cdot \overrightarrow{\boldsymbol{dS}}$

(5) $\boldsymbol{R} = \frac{\Delta V}{\boldsymbol{I}}$

Ex: Consider a quarter-circular washer of rectangular cross section and finite conductivity $\boldsymbol{\sigma}$. Find the resistance if the two electrodes are located at $\boldsymbol{\varphi} = \mathbf{0}$ and $\boldsymbol{\varphi} = \boldsymbol{\pi}/\mathbf{2}$.



$\boldsymbol{BC_1}$	$\boldsymbol{\Phi(0)} = \Delta \boldsymbol{V}$		
$\boldsymbol{BC_2}$	$\boldsymbol{\Phi}\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right) = \mathbf{0}$		

	$\boldsymbol{\Phi} = \boldsymbol{\Phi}(\emptyset)$		
No free charge	$\nabla \cdot \vec{D} = \rho_v = 0$		
	$\nabla \cdot \varepsilon\vec{E} = 0$		
Simple medium	$\varepsilon\nabla \cdot \vec{E} = 0$		
$-\nabla\boldsymbol{\Phi} = \vec{E}$	$\varepsilon\nabla \cdot (-\nabla\boldsymbol{\Phi}) = 0$		
$\nabla \cdot \nabla = \nabla^2$	$-\varepsilon\nabla \cdot \nabla\boldsymbol{\Phi} = -\varepsilon\nabla^2\boldsymbol{\Phi} = 0$		
	$\nabla^2\boldsymbol{\Phi} = 0$		
	$\frac{\partial^2\boldsymbol{\Phi}}{\partial\emptyset^2} = 0$		
	$\frac{\partial\boldsymbol{\Phi}}{\partial\emptyset} = C_1$		
	$\boldsymbol{\Phi} = \emptyset C_1 + C_2$		
Using BCs.,	$\boldsymbol{\Phi}(0) = \Delta V = 0C_1 + C_2$		
	$\boldsymbol{\Phi}\left(\frac{\pi}{2}\right) = 0 = \frac{\pi}{2}C_1 + C_2$		
	$C_1 = \frac{2\Delta V}{\pi}$		
	$C_2 = 0$		
	$\boldsymbol{\Phi} = \frac{2\Delta V}{\pi}\emptyset$		
	$\vec{E} = -\nabla\boldsymbol{\Phi} = -\hat{\emptyset}\frac{2\Delta V}{\pi}\frac{1}{r} == -\frac{2V_0}{\pi}\frac{1}{r}\hat{\emptyset}$		
	$\vec{E} = \vec{E}(r)\hat{\emptyset}$		

	$\vec{J} = \sigma\vec{E} = -\sigma\frac{2\Delta V}{\pi}\frac{1}{r}\hat{\emptyset}$		
	$I = \int \vec{J} \cdot \overrightarrow{dS}$		
	$I = \int_0^h\int_a^b -\sigma\frac{2\Delta V}{\pi} xd\frac{1}{r}\hat{\emptyset} \cdot (-\hat{\emptyset})drdz$		
	$= \left(\int_a^b +\sigma\frac{2\Delta V}{\pi}\frac{1}{r}rdr\right)\left(\int_0^h dz\right)$		
	$= \left(\sigma\frac{2\Delta V}{\pi}\right)\left(\int_a^b \frac{1}{r}rdr\right)\left(\int_0^h dz\right)$		
	$I = \left(\sigma\frac{2\Delta V}{\pi}\right).ln\left(\frac{b}{a}\right).h$		
	$R = \frac{\Delta V}{I}$		
	$R = \frac{\Delta V}{\left(\sigma\frac{2\Delta V}{\pi}\right).ln\left(\frac{b}{a}\right).h}$		
	$R = \left(\left(\sigma\frac{2\Delta V}{\pi}\right).ln\left(\frac{b}{a}\right).h\right)^{-1}$	(Ohm)	