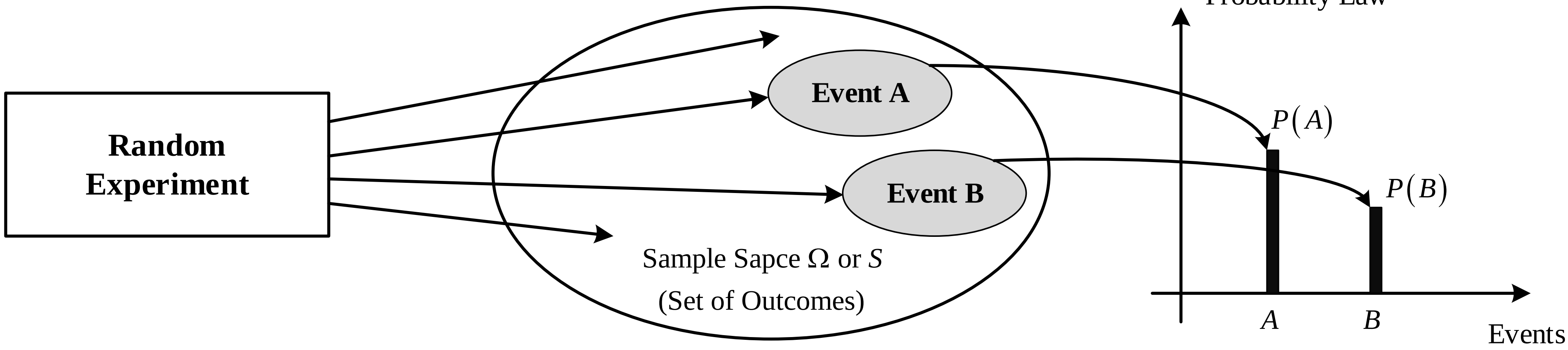
# PROBABILISTIC MODELS

Elements of a Probabilistic Model:

- The sample space Ω
- **The probability law**

  assigns nonnegative numbers to events




Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

# PROBABILISTIC MODELS

## Sample Spaces and Events

- ***Random Experiment***
- ***Trial***
- ***Outcome***
- ***Sample Space S***
- ***Event***
- ***Sure Event***
- ***Impossible Event***

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# PROBABILISTIC MODELS

## Sequential Models

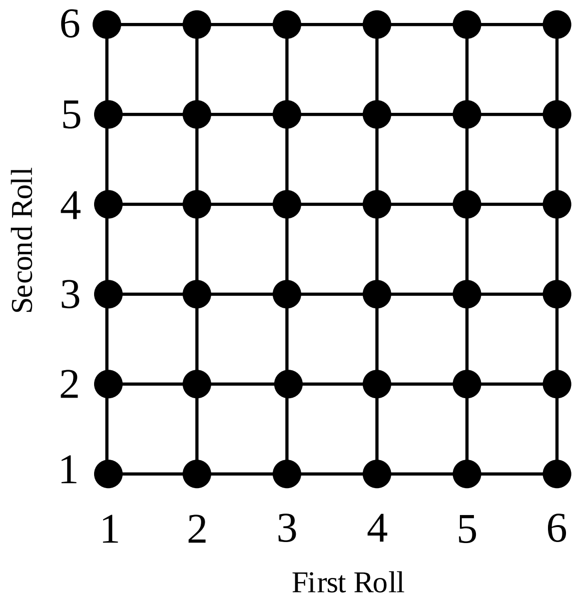
Some experiments have a sequential character

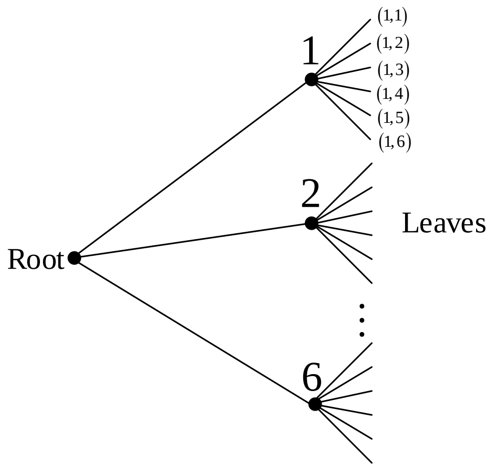
- **Tree-Based Sequential Description**

    **Experiment:** Two rolls of a 6-sided die

# PROBABILISTIC MODELS

## Description of the sample space

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# PROBABILISTIC MODELS

## Probability Axioms

1) **Nonnegativity:**

$P(A) \geq 0$ for every event *A*.

2) **Additivity:**

*A* and *B* are two <u>disjoint events</u>,

$$P(A \cup B) = P(A) + P(B)$$

More generally,

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \cdots$$

3) **Normalization:**

$\Omega$ : universal set

$$P(\Omega) = 1$$

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**PROBABILISTIC MODELS**

Some properties

$$1 = P(\Omega) = P(\Omega \cup \varnothing) = 1 + P(\varnothing),$$

which means

$$P(\varnothing) = 0$$

If $A_1, A_2, \text{and}\, A_3$ disjoint events, then

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1 \cup (A_2 \cup A_3)) \\ &= P(A_1) + P(A_2 \cup A_3) \\ &= P(A_1) + P(A_2) + P(A_3) \end{aligned}$$

# PROBABILISTIC MODELS

**Discrete Models:** Coin tossing, die rolling
**Continuous Models:** Wheel of fortune, between 0 and 1.

**Example 1.2 (textbook)** single coin toss experiment
outcomes: heads ($H$) and tails ($T$).
sample space: $\{H,T\}$
events: $\{H,T\},\{H\},\{T\},\varnothing$

**“equally likely”: equal probabilities**

$$P\left(\{H\}\right)=P\left(\{T\}\right)=0.5$$

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# PROBABILISTIC MODELS
## Example 1.2. (Continued)

Additivity and normalization axioms

$$P\left(\{H,T\}\right)=P\left(\{H\}\right)+P\left(\{T\}\right)=1$$

Nonnegativity axiom

$$P\left(\text{"Any Event"}\right)\geq 0$$

**The probability law:**

$$P\left(\{H,T\}\right)=1,\qquad P\left(\{H\}\right)=0.5,\qquad P\left(\{T\}\right)=0.5,\qquad P\left(\varnothing\right)=0,$$

# PROBABILISTIC MODELS

## Discrete Probability Law:

**finite number of possible outcomes**

$$P\left(\left\{s_1, s_2, \ldots, s_n\right\}\right) = P\left(s_1\right) + P\left(s_2\right) + \cdots + P\left(s_n\right)$$

## Uniform Probability Law:

- *n* possible outcomes, equally likely

$$P\left(A\right) = \frac{\text{number of elements of } A}{n}$$

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# PROBABILISTIC MODELS

## Example: Discrete Sample Space

Single roll of a 6-sided die.

- Discrete Sample Space:

$$S = \{1, 2, 3, 4, 5, 6\}$$

- Some Events:

$A = \{\text{a number less than 4 appears}\} = \{1, 2, 3\}$

$A = \{\text{an even number appears}\} = \{2, 4, 6\}$

$A = \{\text{1 appears}\} = \{1\}$

$A = \{\text{a number greater than 6 appears}\} = \varnothing$

$A = \{\text{1 or 2 appears}\} = \{1, 2\}$

$A = \{\text{1 and 2 appears}\} = \varnothing$

$A = \{\text{an integer in the interval [1,6] appears}\} = \{1, 2, 3, 4, 5, 6\}$

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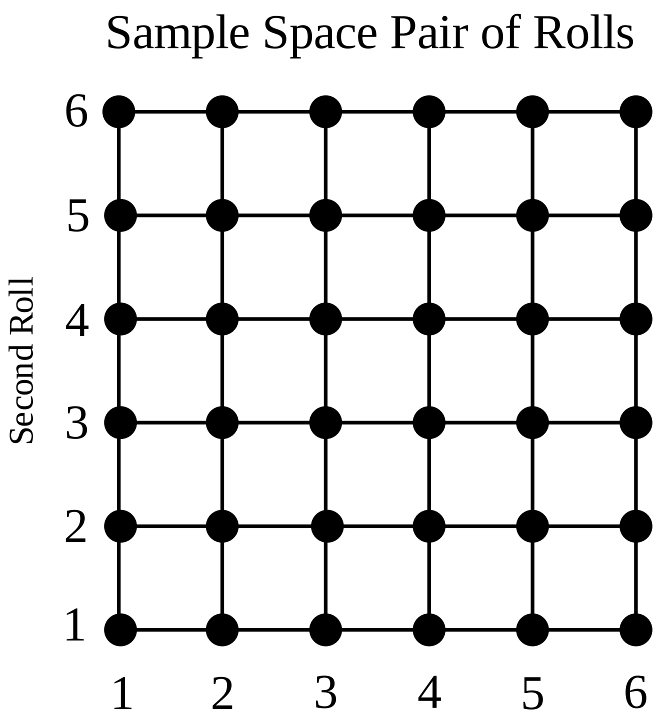
# PROBABILISTIC MODELS

**Example:** Consider the experiment of rolling a pair of <u>fair dice</u>.

**<u>Fair dice assumption:</u> probability each outcome: 1/36.**

$$S = \{(i, j) : (1,1), (1,2), (1,3), \ldots, (4,3), \ldots, (6,4), (6,5), (6,6)\}$$




$$P(\{\text{the sum is even}\}) = \frac{18}{36} = \frac{1}{2}$$

$$P(\{\text{the sum is odd}\}) = \frac{18}{36} = \frac{1}{2}$$

$$P(\{\text{the first roll is larger than the second one}\}) = \frac{1}{3}$$

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# PROBABILISTIC MODELS
## Continuous Models

**Example 1.4.** A wheel of fortune, continuously calibrated [0,1]
Experiment: a single spin, numbers in the interval $\Omega = [0,1]$.
fair wheel: equally likely,
What is the probability of the event consisting of a single element?

**Answer:** It cannot be positive. Remember the additivity axiom,
the probability of a single element must be 0.

**What is the probability of any subinterval $[a,b]$?**

$$P([a,b]) = \frac{\text{Length of Interested Interval}}{\text{Length of Sample Space}} = \frac{b-a}{1} \quad \rightarrow \quad \text{Continuous Uniform Law}$$

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# PROBABILISTIC MODELS

## Properties of Probability Laws:

1) If $A \subset B$, then $P(A) \leq P(B)$

2) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

3) $P(A \cup B) \leq P(A) + P(B)$

4) $P(A \cup B \cup C) = P(A) + P(A^C \cap B) + P(A^C \cap B^C \cap C)$

Proof: **(homework)**

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