CONDITIONAL PROBABILITY

CONDITIONAL PROBABILITY

Knowledge: the event B has occurred
$P(A|B) = ?$

We have a **<u>reduced sample space *B*.</u>**

A occurs if the outcome is in $A \cap B$.

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONAL PROBABILITY

Definition

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}, \qquad P(B) > 0$$

- **a probability law: a new universe *B*.**
- outcomes: finitely many and equally likely

$$P(A|B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B}.$$

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CONDITIONAL PROBABILITY

Do Conditional Probabilities Specify a Legitimate Probability Law?

- $P(A \cap B) \geq 0,\ P(B) > 0 \quad \Rightarrow \quad P(A \mid B) > 0$, non-negativity axiom

- $P(\Omega \mid B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$, normalization axiom

CONDITIONAL PROBABILITY

- Verification of the additivity axiom for conditional probability

$$P(A_1 \cup A_2 \mid B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 \mid B) + P(A_2 \mid B)$$

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CONDITIONAL PROBABILITY

Example 1.6 (textbook) tossing a fair coin three times.

A= {more heads than tails come up}, B= {first toss is a head}

Find $P(A|B)$.

$$S = \{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{THH}, \mathrm{TTH}, \mathrm{THT}, \mathrm{HTT}, \mathrm{TTT}\}$$

$$A = \{HHH, HHT, HTH, THH\} \to \quad P(A) = 4/8 = 1/2$$

$$B = \{HHH, HHT, HTH, HTT\} \to \quad P(B) = 4/8 = 1/2$$

$$A \cap B = \{HHH, HHT, HTH\} \to \quad P(A \cap B) = 3/8$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{4/8} = \frac{3}{4} \qquad \text{or}$$

$$P(A|B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B} = \frac{3}{4}$$

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CONDITIONAL PROBABILITY

Example Rolling a 6-sided die.

A= {number 1 appears}, B= {an odd number appears}, C= {number 1 or 2 appears}

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

$$P(\mathrm{C} \mid B) = \frac{P(C \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

$$P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{1/6}{2/6} = \frac{1}{2}$$

HW: Example 1.7, Example 1.8 (textbook)

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONAL PROBABILITY

Modeling using Conditional Probability

Example 1.9 (textbook)

Let we define the events A and B such that

$$A = \{\text{an aircraft is present}\},\ B = \{\text{the radar generates an alarm}\}$$
$$A^C = \{\text{an aircraft is not present}\},\ B^C = \{\text{the radar does not generate an alarm}\}$$

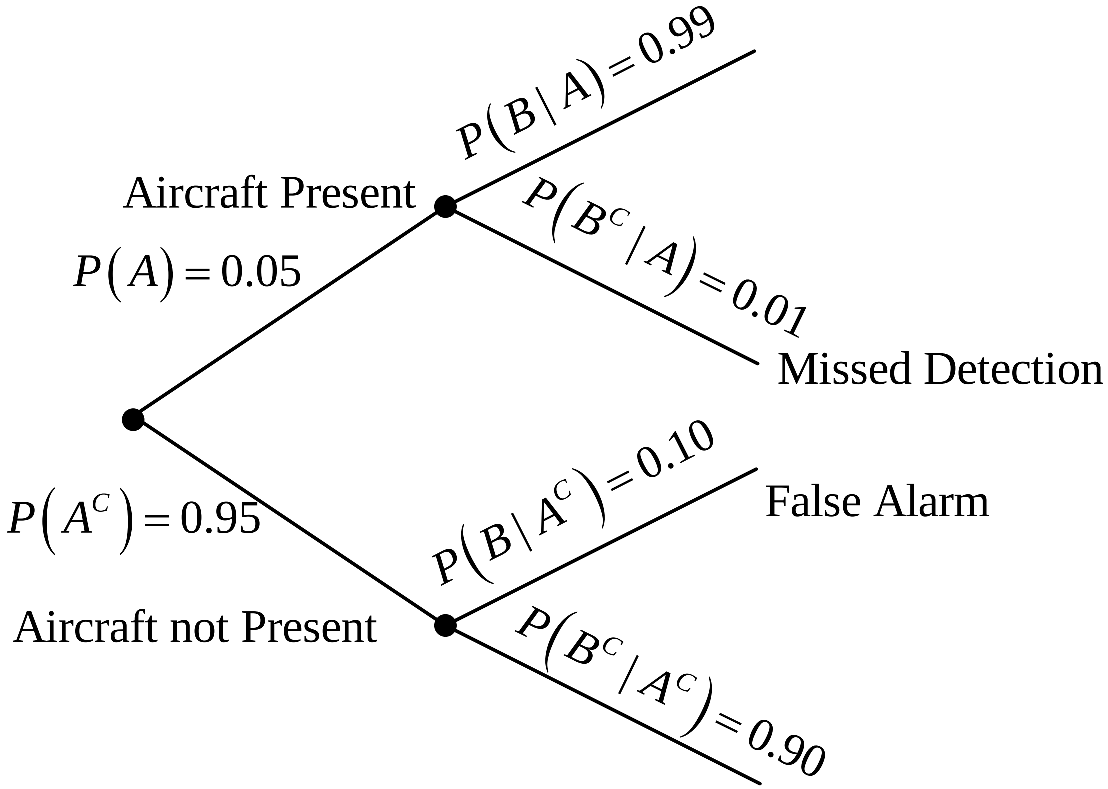
Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONAL PROBABILITY

Example 1.9. Radar Detection (Continued)

$$P(\text{False alarm}) = P(\text{not present and alarm}) = P\left(A^C \cap B\right) = P\left(A^C\right)P\left(B \mid A^C\right) = 0.95 \times 0.10 = 0.095$$

$$P(\text{Missed detection}) = P(\text{present and no detection}) = P\left(A \cap B^C\right) = P(A)P\left(B^C \mid A\right) = 0.05 \times 0.01$$
$$= 0.0005$$



Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONAL PROBABILITY
Multiplication Rule

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)\ldots P\left(A_n \mid \bigcap_{i=1}^{n-1} A_i\right)$$

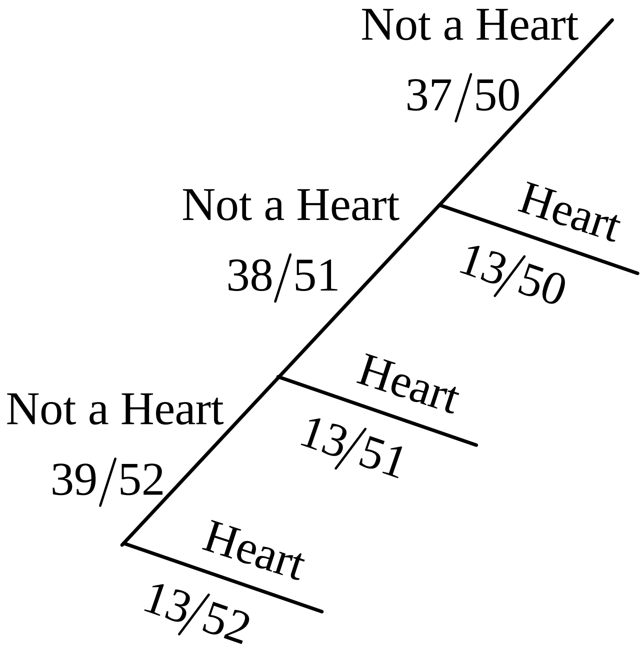
Verification of the multiplication rule:

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\frac{P(A_1 \cap A_2)}{P(A_1)}\frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)}\cdots\frac{P\left(\bigcap_{i=1}^{n} A_i\right)}{P\left(\bigcap_{i=1}^{n-1} A_i\right)}$$

$$= P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)\ldots P\left(A_n \mid \bigcap_{i=1}^{n-1} A_i\right)$$

CONDITIONAL PROBABILITY
Multiplication Rule

Example 1.10 (textbook) Three cards are drawn from an 52-card deck, no replacement, equally likely. Find the probability that none of the three cards is a heart.



$A_i = \{\text{the } i\text{th card is not a heart}\}, \quad i = 1,2,3$

We want to calculate $P(A_1 \cap A_2 \cap A_3)$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)$$

$$P(A_1) = \frac{39}{52}, \quad P(A_2 \mid A_1) = \frac{38}{51}, \quad P(A_3 \mid A_1 \cap A_2) = \frac{37}{50}$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) = \frac{39}{52} \times \frac{38}{51} \times \frac{37}{50}$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.