TOTAL PROBABILITY THEOREM AND BAYES' RULE

Total Probability Theorem:

$A_1, A_2, ..., A_n$ <u>form a **partition** of the set *S*</u>.

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$
$$= P(A_1)P(B | A_1) + P(A_2)P(B | A_2) + \cdots + P(A_n)P(B | A_n)$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

TOTAL PROBABILITY THEOREM AND BAYES' RULE
Total Probability Theorem

"Divide-and-Conquer" approach

- Divide the sample space
- calculate P(B) as the weighted average

➢ key: choosing an appropriate partition $A_1,\ldots,A_n$.

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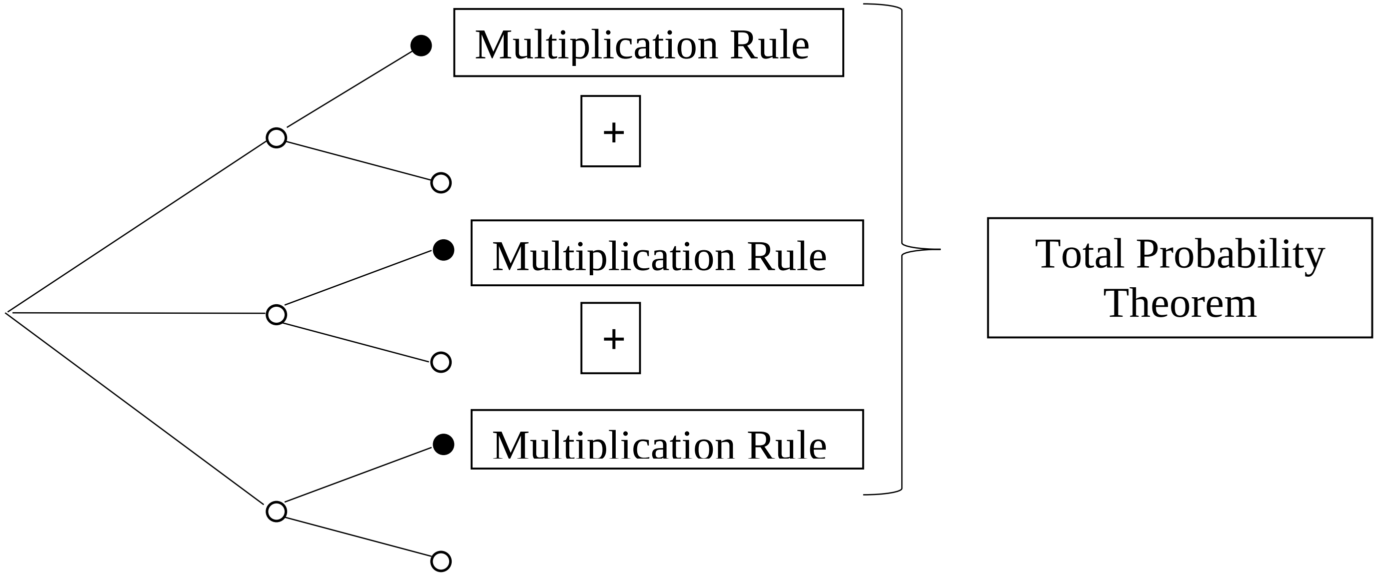
TOTAL PROBABILITY THEOREM AND BAYES' RULE
Total Probability Theorem



Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

TOTAL PROBABILITY THEOREM AND BAYES' RULE

Example 1.13 (**textbook**) chess tournament

A_i: playing with an opponent of type i

$$P(A_1) = 0.5, \quad P(A_2) = 0.25, \quad P(A_3) = 0.25$$

B: winning

$$P(B \mid A_1) = 0.3, \quad P(B \mid A_2) = 0.4, \quad P(B \mid A_3) = 0.5$$

$$\begin{aligned} P(B) &= P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3) \\ &= 0.5 \times 0.3 + 0.25 \times 0.4 + 0.25 \times 0.5 \\ &= 0.375 \end{aligned}$$

HW: Example 1.15

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

TOTAL PROBABILITY THEOREM AND BAYES' RULE

Bayes' Rule

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(B)}$$

$$= \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)}$$

$$= \frac{P(A_i)P(B \mid A_i)}{\sum_{i=1}^{n} P(A_i)P(B \mid A_i)}$$

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TOTAL PROBABILITY THEOREM AND BAYES' RULE

Bayes' Rule

$P(A_i)$: a priori probability of event A_i

(Probability of event A_i without knowing event B has occurred.)

$P(A_i \mid B)$: a posteriori probability of event A_i

(Probability of event A_i knowing event B has occurred.)

Verification

$$P(A_i \cap B) = P(A_i)P(B \mid A_i) = P(B)P(A_i \mid \mathrm{B}) \quad \Rightarrow \quad P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(B)}$$

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(B)} == \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)}$$

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TOTAL PROBABILITY THEOREM AND BAYES' RULE

Example 1.16 (textbook)

$A = \{\text{an aircraft is present}\}$, $B = \{\text{the radar generates an alarm}\}$

$A^C = \{\text{an aircraft is not present}\}$, $B^C = \{\text{the radar does not generate an alarm}\}$

We are given that $P(A) = 0.05$, $P(B \mid A) = 0.99$, $P(B \mid A^C) = 0.10$

Applying Bayes' rule with $A_1 = A$ and $A_2 = A^C$

$$P(\text{aircraft present}|\text{alarm}) = P(A \mid B)$$

$$= \frac{P(A)P(B \mid A)}{P(B)}$$

$$= \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A^C)P(B \mid A^C)}$$

$$= \frac{0.05 \times 0.99}{0.05 \times 0.99 + 0.95 \times 0.1} \approx 0.3426 \qquad !!!$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

TOTAL PROBABILITY THEOREM AND BAYES' RULE

Example 1 .17 (textbook) chess problem of Example 1.13

A_i is the event of getting an opponent of type *i*

$$\mathrm{P}(A_1) = 0.5, \quad \mathrm{P}(A_2) = 0.25, \quad \mathrm{P}(A_3) = 0.25$$

B: winning: $\mathrm{P}(B \mid A_1) = 0.3, \quad \mathrm{P}(B \mid A_2) = 0.4, \quad \mathrm{P}(B \mid A_3) = 0.5$

Assume that you win. $\mathrm{P}(A_1 \mid B)$?

Using Bayes' rule,

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)}$$

$$= \frac{0.5 \times 0.3}{0.5 \times 0.3 + 0.25 \times 0.4 + 0.25 \times 0.5}$$

$$= 0.4$$

TOTAL PROBABILITY THEOREM AND BAYES' RULE

Example 1.18 (textbook)

A = {the person has the disease}
B = {the test results are positive}

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A^c)P(B \mid A^c)}$$

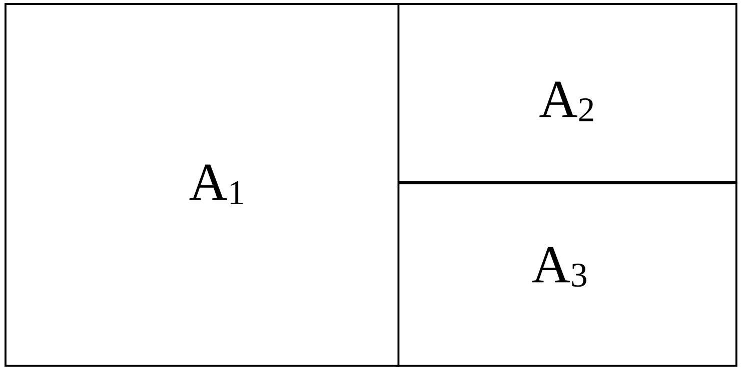
$$= \frac{0.001 \times 0.95}{0.001 \times 0.95 + 0.999 \times 0.05}$$

$$= 0.0187$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

Overview- Total Probability Theorem, Multiplication Rule and Bayes' Rule

The key: choose appropriately the partition



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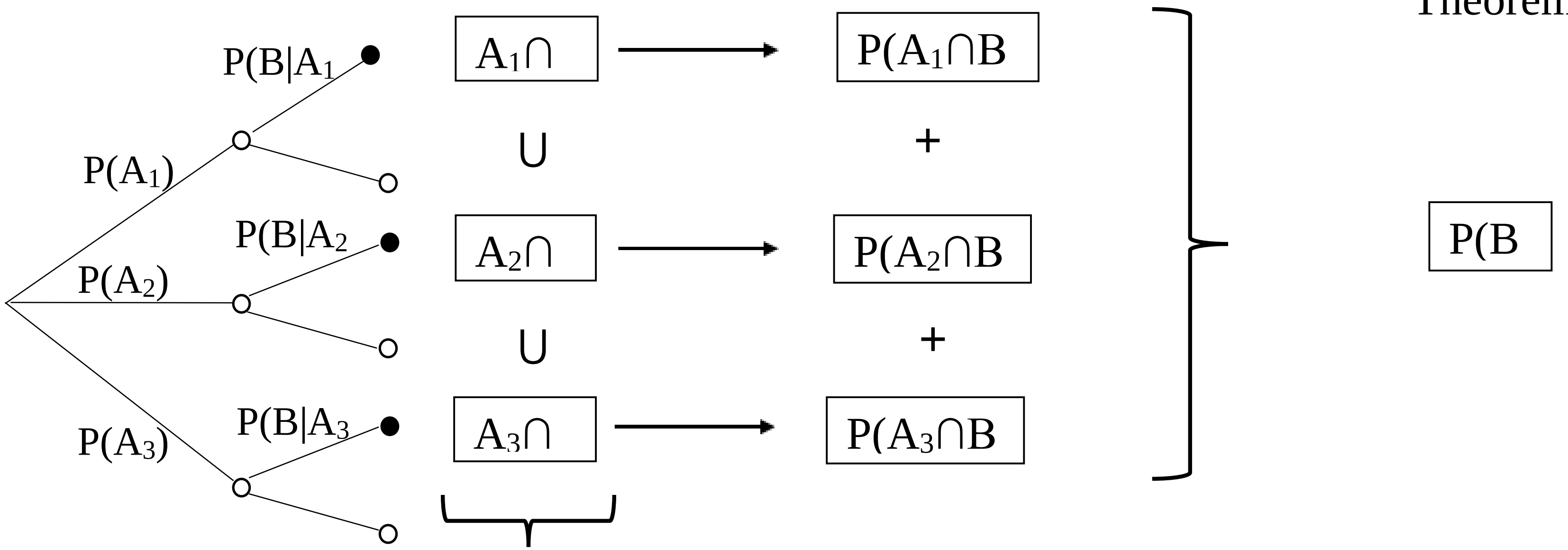
Overview-(continued)

EVENTS PROBABILITIES (obtained by Multiplication Rule) PROBABILITY (obtained by Total Probability Theorem)



EVENT: **B** = $(A_1 \cap B) \cup (A_2 \cap B) \cup$

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(B)}$$

BAYES' RULE

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.