INDEPENDENCE

If

$$P(A|B) = P(A)$$

then it is said that *A* and *B* are independent events.

By using $P(A|B) = P(A \cap B)/P(B)$, independence can be given as

$$P(A \cap B) = P(A)P(B)$$

Symmetric property:

If *A* is independent of *B*, then *B* is independent of *A*,
Therefore it is said that *A* and *B* are **independent events**.

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

INDEPENDENCE

easy to understand intuitively, but it **is not easily visualized**.

Note that two disjoint events A and B with $P(A)>0$ and $P(B)>0$ cannot be **independent**, $P(A \cap B)=0$

Example 1.19 (textbook) Experiment: two successive rolls of a 4-sided die, equally likely.

Are the events

$A=\{\text{1st roll is a 1}\}, \quad B=\{\text{sum of the two rolls is a 5}\}$

independent?

$$P(A \cap B)=P\left(\text{the result of the two roll is } (1,4)\right)=\frac{1}{16}$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

INDEPENDENCE

Example 1.19 (textbook)-continued

$$P(A) = \frac{\text{number of elements of } A}{\text{total number of possible outcomes}} = \frac{4}{16}$$

$$P(B) = \frac{\text{number of elements of } B}{\text{total number of possible outcomes}} = \frac{4}{16},$$

$P(A \cap B) = P(A)P(A)$ and the events A and B are independent.

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

Conditional Independence

$$P\left(A \cap B \mid C\right) = P\left(A \mid C\right) P\left(B \mid C\right)$$

Example 1.20 (textbook) two independent fair coin tosses, equally likely

$H_1 = \{\text{1st toss is a head}\}$, $H_2 = \{\text{2nd toss is a head}\}$,

$D = \{\text{the two tosses have different results}\}$

H_1 and H_2 are unconditionally independent. However,

$$P\left(H_1 \mid D\right) = \frac{1}{2}, \quad P\left(H_2 \mid D\right) = \frac{1}{2}, \quad P\left(H_1 \cap H_2 \mid D\right) = 0 \quad ,$$

universe:{HT,TH}

$$P\left(H_1 \cap H_2 \mid D\right) \neq P\left(H_1 \mid D\right) P\left(H_2 \mid D\right)$$

H_1, H_2 are not conditionally independent.

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

Independence of a Collection of Events

Events: A1 , A2 , and A3

conditions for independence:

$$P(A_1 \cap A_2) = P(A_1)P(A_2),$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3),$$

$$P(A_2 \cap A_3) = P(A_2)P(A_3),$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3).$$

Examples 1.22 and 1.23 (textbook, homework).

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

Independent Trials and the Binomial Probabilities

Independent Bernoulli trials:

Independent Trials: independent but identical stages
Bernoulli Trials: only two possible results,

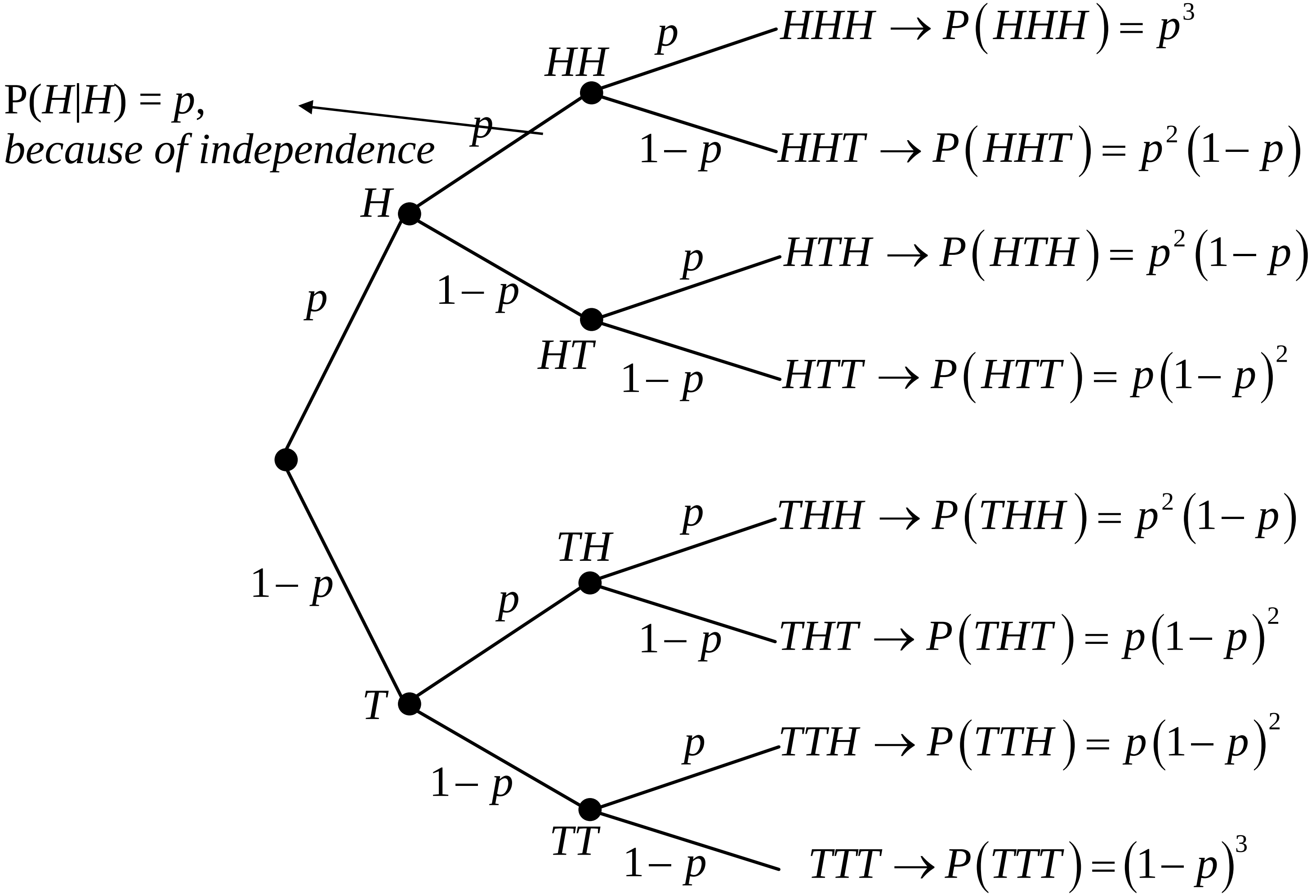
Example (textbook p.41-43):
Experiment: n independent tosses of a **biased** coin,
The probability of "heads" : p
The probability of "tails" : 1-p.

Events $A_1, A_2, \ldots, A_n$ are independent,

$$A_i = \{i\text{th toss is a head}\}$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

Example (textbook p.41-43):



Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

Independent Trials and Binomial Probabilities

$$\mathrm{P}\left(\text{"}k \text{ heads in any particular 3-long sequence"}\right) = p^k (1-p)^{3-k}$$

Generalization for n tosses:

$$\mathrm{P}\left(\text{"}k \text{ heads in any particular } n\text{-long sequence"}\right) = p^k (1-p)^{n-k}$$

The probability

$$p(k) = \mathrm{P}\left(\text{"}k \text{ heads in an } n\text{-toss sequence"}\right)$$

Independent Trials and Binomial Probabilities

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k} \rightarrow \text{Binomial Probability Law}$$

$\binom{n}{k}$: number of distinct n-toss sequences contain k heads. (read as "n choose k")

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

Independent Trials and Binomial Probabilities

The numbers $\binom{n}{k}$ are known as the **binomial coefficients.**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad k = 0, 1, 2, \ldots, n$$

where for any positive integers i we have

$$i! = 1 \times 2 \times 3 \times \cdots \times i, \qquad 0! = 1$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.