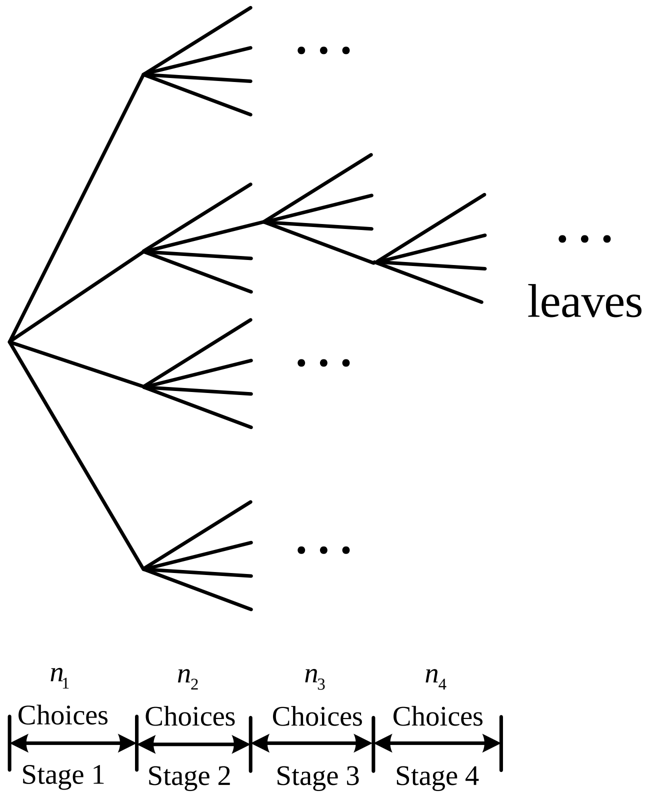
COUNTING

The calculation of probabilities often involves counting the number of outcomes in various events.

The Counting Principle: based on **a divide-and-conquer approach**,



Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

Counting

The Counting Principle

Consider r stages:

- n_1 possible results for the first stage.
- For each result of the first stage, there are n_2 possible results at the second stage.
- Generally, for each results of the first $i-1$ stage, there are n_i possible results at the ith stage.
- The total number of results:

$$n_1 \times n_2 \times \cdots \times n_r$$

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Counting

Example 1.26 (textbook). A telephone number: 7-digit sequence, the first digit cannot be 0 or 1. How many distinct telephone numbers are there?

$$8\times 10\times\cdots\times 10 = 8\times 10^6$$

Example 1.27 (textbook) How many subsets does the set $\{s_1, s_2, \ldots, s_n\}$ have?

$$\underbrace{2\times 2\times\cdots\times 2}_{n \text{ times}} = 2^n.$$

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Counting

Permutation: Selection of *k* objects out of the *n* objects, paying attention to order of the selection.

***k*-permutations:**
Selecting *k* of *n* distinct object

How many different ways are there (order of the selection matters)?

By the Counting Principle, *k*-permutations

$$n(n-1)\cdots(n-k+1)=\frac{n\times(n-1)\times\cdots\times(n-k+1)\times(n-k)\times\cdots\times 2\times 1}{(n-k)\times\cdots\times 2\times 1}=\frac{n!}{(n-k)!}$$

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Counting

Example taking two of the four letters A, B, C, and D.

2-permutations:

AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC.

$$\frac{n!}{(n-k)!} = \frac{4!}{2!} = 4 \times 3 = 12$$

Example 1.28 (textbook) The number of words that consist of four distinct letters

4-permutations of the 26 letters

$$\frac{n!}{(n-k)!} = \frac{26!}{22!} = 26 \times 25 \times 24 \times 23 = 358,800$$

Example 1.29. (textbook) homework

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Counting

Combination: Selection of k objects out of the n objects, without paying any attention to order of the selection

2-permutations of A, B, C, and D

AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC.

Combinations of two out four of these letters

AB, AC, AD, BC, BD, CD

The number of possible combinations :

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

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Counting

Example 1.30 (textbook). combinations of two out of the four letters A,B,C, and D:

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$$

Counting

Partitions:

Nonnegative integers $n_1, n_2, \ldots, n_r$

$$n_1 + n_2 + \ldots + n_r = n$$

Partitions of the set into r disjoint subsets. *n* objects is divided into *r* disjoint groups, there are n_i elements of *i*th group. The number **<u>ways for dividing *n* objects into *r* disjoint groups?</u>**

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Counting

Partitions and Multinomial Coefficients:

○ Use the Counting Principle

$$\binom{n}{n_1}\times\binom{n-n_1}{n_2}\times\binom{n-n_1-n_2}{n_3}\times\cdots\times\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$

which is equal to

$$\frac{n!}{n_1!(n-n_1)!}\times\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\times\cdots\times\frac{(n-n_1-\cdots-n_{r-1})!}{n_r!(n-n_1-\cdots-n_r)!}=\frac{n!}{n_1!\mathrm{n}_2!\cdots\mathrm{n}_r!}=\underbrace{\binom{n}{n_1,n_2,\ldots,n_r}}_{\text{Multinomial Coefficient}}$$

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Counting

Example 1.33 (textbook). There are 4 graduate and 12 undergraduate students in a class. Divide the class into four groups of 4 **randomly**. What is the probability that each group includes a graduate student?

A typical outcome: partitioning the 16 students into four groups of 4.

$$N_S = \binom{16}{4,4,4,4} = \frac{16!}{4!4!4!4!}$$

- Distributing four graduate students to the four group, number of ways:4!
- Distributing remaining 12 undergraduate students to the four groups, number of ways:

$$\binom{12}{3,3,3,3} = \frac{12!}{3!3!3!3!}$$

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Counting

Example 1.33 (textbook)

Total number of elements

$$N_A = 4! \times \frac{12!}{3!3!3!3!}$$

The probability of the event is

$$P(A) = \frac{N_A}{N_S} = \frac{4! \times \dfrac{12!}{3!3!3!3!}}{\dfrac{16!}{4!4!4!4!}}$$

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