FUNCTIONS OF RANDOM VARIABLES

Linear function of *X*

$$Y = g(X) = aX + b$$

where *a* and *b* are scalars.

Nonlinear function of *X:*

$$Y = g(X) = \log(X)$$

Y is also a **random variable**

PMF of *Y* can $$p_Y(y) = \sum_{\{x | g(x) = y\}} p_X(x)$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

FUNCTIONS OF RANDOM VARIABLES

Example 2.1 (textbook). PMF of X

$$p_X(x) = \begin{cases} 1/9 & \text{if } x \text{ is an integer in the range } [-4, 4] \\ 0 & \text{otherwise} \end{cases}$$

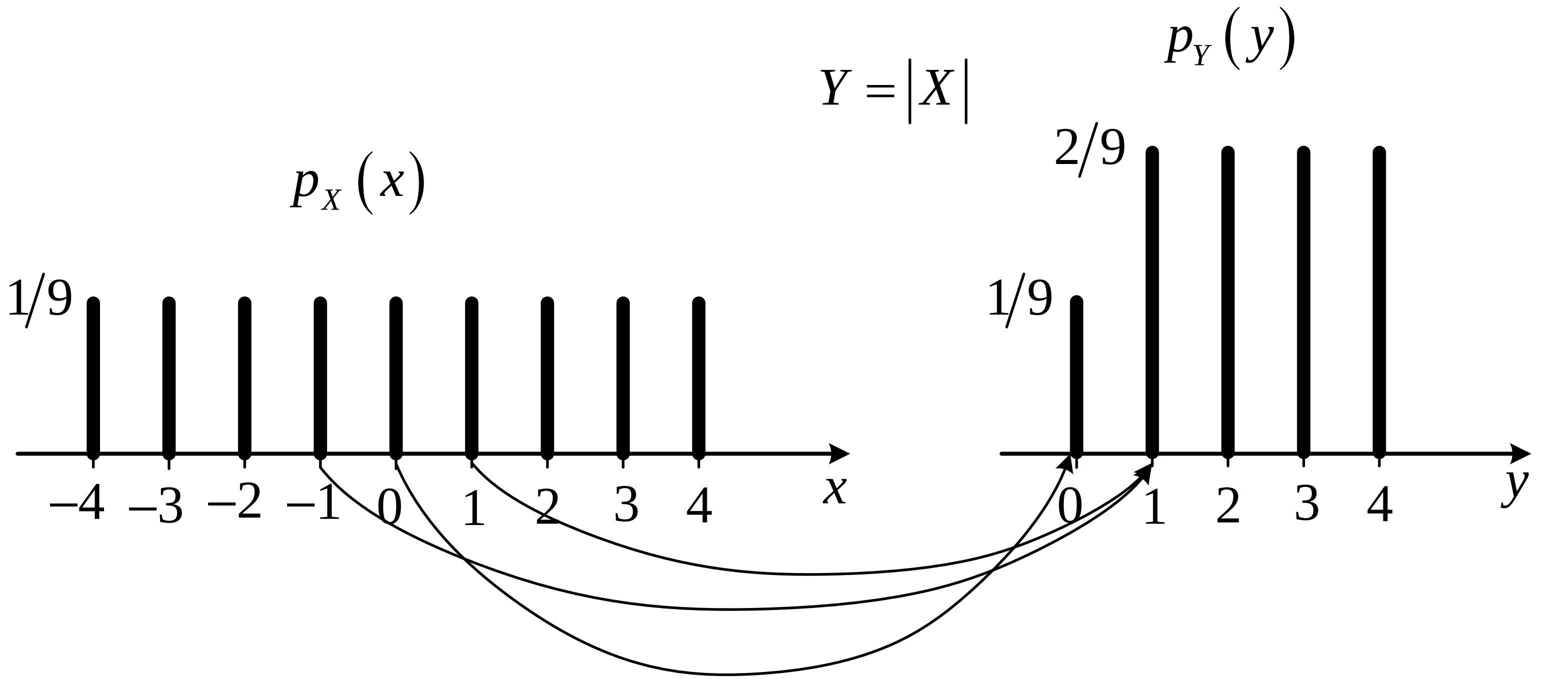
Let $Y = |X|$. Find the PMF of Y.

The possible values for Y: $y = 0, 1, 2, 3, 4$, i.e., $S_Y = \{0, 1, 2, 3, 4\}$.

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

FUNCTIONS OF RANDOM VARIABLES

Example 2.1 (textbook). (Continued)



$$p_Y(0) = \sum_{\{x \mid |x| = y\}} p_X(x) = p_X(0) = \frac{1}{9}$$

$$p_Y(1) = \sum_{\{x \mid |x| = y\}} p_X(x) = p_X(-1) + p_X(1) = \frac{2}{9}$$

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FUNCTIONS OF RANDOM VARIABLES

Example 2.1 (textbook). (Continued)

$$p_Y(2) = \sum_{\{x \mid |x| = y\}} p_X(x) = p_X(-2) + p_X(2) = \frac{2}{9}$$

$$p_Y(3) = \sum_{\{x \mid |x| = y\}} p_X(x) = p_X(-3) + p_X(3) = \frac{2}{9}$$

$$p_Y(4) = \sum_{\{x \mid |x| = y\}} p_X(x) = p_X(-4) + p_X(4) = \frac{2}{9}$$

$$p_Y(y) = \begin{cases} 2/9, & \text{if } y = 1, 2, 3, 4, \\ 1/9, & \text{if } y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

FUNCTIONS OF RANDOM VARIABLES

Example 2.1. (Continued)

Consider the random variable $Z = X^2$.

PMF of Z :

$$p_Z(z) = \sum_{\{x|x^2=z\}} p_X(x) = \sum_{\{y|y^2=z\}} p_Y(y)$$

The possible values of Z : $z = 0,1,4,9,16$ and its PMF is

$$p_Z(z) = \begin{cases} 2/9, & \text{if } z = 1,4,9,16, \\ 1/9, & \text{if } z = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

EXPECTATION, MEAN AND VARIANCE

Expectation

- Summarizes the information about *X* in a single representative number.

- Weighted average of the possible values of *X*.

Definition: The **expectation** (or the **mean**) of a discrete random variable *X*

$$E[X] = \sum_x x p_X(x)$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

EXPECTATION, MEAN AND VARIANCE

Example 2.2 (textbook).
Experiment: two independent coin tosses, probability of a head : ¾
X : the number of heads obtained. E[*X*]=?

A binomial random variable, n = 2 and p = 3/4. Its PMF

$$p_X(k) = \begin{cases} (1/4)^2, & \text{if } k = 0, \\ 2.(1/4)(3/4), & \text{if } k = 1, \\ (3/4)^2, & \text{if } k = 2. \end{cases}$$

$$E[X] = \sum_{k \in \{0,1,2\}} k p_X(k) = \sum_{k=0}^{2} k p_X(k) = 0 \times \frac{1}{16} + 1 \times \frac{6}{16} + 2 \times \frac{9}{16} = \frac{24}{16} = \frac{3}{2}$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

EXPECTATION, MEAN AND VARIANCE
Variance, Moments, and the Expected Value Rule

Variance

$$\mathrm{var}(X) = E\left[\left(X - E[X]\right)^2\right]$$

- is always nonnegative.
- provides a measure of dispersion of X around its mean.
- **Standard deviation** of X

$$\sigma_X = \sqrt{\mathrm{var}(X)}$$

The **standard deviation** has the same units as X.

EXPECTATION, MEAN AND VARIANCE

Variance, Moments, and the Expected Value Rule

$E[X] = \sum_x x p_X(x)$ → First moment of X

$E[X^2] = \sum_x x^2 p_X(x)$ → Second moment of X

$E[X^3] = \sum_x x^3 p_X(x)$ → Third moment of X

$\vdots$

$E[X^n] = \sum_x x^n p_X(x)$ → nth moment of X

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

EXPECTATION, MEAN AND VARIANCE

Variance, Moments, and the Expected Value Rule

Example 2.3 (textbook) Consider the random variable X of Example 2.1

$$p_X(x) = \begin{cases} 1/9 & \text{if } x \text{ is an integer in the range } [-4, 4] \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_x x p_X(x) = \frac{1}{9} \sum_{x=-4}^{4} x = 0$$

or simply $E[X] = 0$ since $p_X(x)$ is symmetric around 0.

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

EXPECTATION, MEAN AND VARIANCE
Variance, Moments, and the Expected Value Rule

Example 2.3 (Continued)

$$\text{Let } Z = \left(X - E[X]\right)^2 = X^2$$

$$p_Z(z) = \begin{cases} 2/9 & z = 1, 4, 9, 16 \\ 1/9 & z = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{var}(X) = E[Z] = \sum_z z p_Z(z) = 0 \cdot \frac{1}{9} + 1 \cdot \frac{2}{9} + 4 \cdot \frac{2}{9} + 9 \cdot \frac{2}{9} + 16 \cdot \frac{2}{9} = \frac{60}{9}$$

EXPECTATION, MEAN AND VARIANCE
Variance, Moments, and the Expected Value Rule
Expected Value Rule for Functions of Random Variables

$$E\left[g(x)\right] = \sum_x g(x)\, p_X(x)$$

Note that** $g(x)$ **need not be linear.

Using this rule, we can obtain the variance of *X* as

$$\mathrm{var}(X) = E\left[\left(X - E[X]\right)^2\right] = \sum_x \left(x - E[X]\right)^2 p_X(x)$$

Similarly, the *n*th moment is given by

$$E\left[X^n\right] = \sum_x x^n p_X(x)$$

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