JOINT PMFS OF MULTIPLE R.V.s

For two discrete random variables X and Y associated with the same experiment, the **joint PMF** of X and Y can be given as

$$p_{X,Y}(x,y) = p_{XY}(x,y) = P(X = x, Y = y)$$

Let A be a set of all pairs (x,y), then

$$P((X,Y) \in A) = \sum_{(x,y)\in A} p_{X,Y}(x,y)$$

Note that fo universal set:

$$P((X,Y) \in S_{XY}) = \sum_{(x,y)\in S_{XY}} p_{X,Y}(x,y) = \sum_{x}\sum_{y} p_{X,Y}(x,y) = 1$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

JOINT PMFs OF MULTIPLE R.V.s

The **marginal PMFs** of *X* and *Y* are:

$$p_X(x) = \sum_y p_{X,Y}(x, y), \qquad p_Y(y) = \sum_x p_{X,Y}(x, y)$$

Tabular Method

$$p_X(x) = \sum_y p_{X,Y}(x, y), \qquad p_Y(y) = \sum_x p_{X,Y}(x, y)$$

See Example 2.9 (textbook)

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

JOINT PMFs OF MULTIPLE R.V.s
Functions of Multiple Random Variables

$$Z = g(X, Y)$$

defines a new random variable. Its PMF can be obtained by using the joint PMF $p_{X,Y}(x, y)$

$$p_Z(z) = \sum_{\{(x,y) | g(x,y)=z\}} p_{X,Y}(x, y)$$

The extension of the expected value rule

$$E[Z] = E[g(X, Y)] = \sum_x \sum_y g(x, y) \, p_{X,Y}(x, y)$$

If g is **linear** of the form $aX + bY + c$, then

$$E[g(X, Y)] = E[aX + bY + c] = aE[X] + bE[Y] + c$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

JOINT PMFs OF MULTIPLE R.V.s

Example 2.9 – continued- (textbook):

Consider the random variables *X* and *Y* whose joint PMF is given in Fig. 2.10
Let *Z* be

$$Z = X + 2Y$$

The PMF of Z

$$p_Z(z) = \sum_{\{(x,y)|x+2y=z\}} p_{X,Y}(x,y)$$

$$p_Z(3) = \sum_{(x,y)\in\{(1,1)\}} p_{X,Y}(x,y) = p_{X,Y}(1,1) = \frac{1}{20}$$

$$p_Z(4) = \sum_{(x,y)\in\{(2,1)\}} p_{X,Y}(x,y) = p_{X,Y}(2,1) = \frac{1}{20}$$

$$p_Z(5) = \sum_{(x,y)\in\{(1,2),(3,1)\}} p_{X,Y}(x,y) = p_{X,Y}(1,2) + p_{X,Y}(3,1) = \frac{1}{20} + \frac{1}{20} = \frac{2}{20}$$

…

…

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

JOINT PMFs OF MULTIPLE R.V.s

Example 2.9 (continued):

$$E[Z] = \sum_z z p_Z(z) = 3 \cdot \frac{1}{20} + 4 \cdot \frac{1}{20} + 5 \cdot \frac{2}{20} + 6 \cdot \frac{2}{20} + 7 \cdot \frac{4}{20}$$

$$+ 8 \cdot \frac{3}{20} + 9 \cdot \frac{3}{20} + 10 \cdot \frac{2}{20} + 11 \cdot \frac{1}{20} + 12 \cdot \frac{1}{20}$$

$$= 7.55$$

Alternatively,

$$E[Z] = E[X] + 2E[Y]$$

By using the marginal PMFs

$$E[X] = 1 \cdot \frac{3}{20} + 2 \cdot \frac{6}{20} + 3 \cdot \frac{8}{20} + 4 \cdot \frac{3}{20} = \frac{51}{20}$$

$$E[Y] = 1 \cdot \frac{3}{20} + 2 \cdot \frac{7}{20} + 3 \cdot \frac{7}{20} + 4 \cdot \frac{3}{20} = \frac{50}{20}$$

$$E[Z] = \frac{51}{20} + 2 \cdot \frac{50}{20} = 7.55$$

JOINT PMFs OF MULTIPLE R.V.s

More Than Two R.V.s

The joint PMF of *X*, *Y*, and *Z* is defined as

$$p_{X,Y,Z}(x,y,z)=P(X=x,Y=y,Z=z)$$

for all numerical values (x,y,z). Corresponding marginal PMFs can be obtained as:

$$p_{X,Y}(x,y)=\sum_{z} p_{X,Y,Z}(x,y,z), \qquad p_{X}(x)=\sum_{y}\sum_{z} p_{X,Y,Z}(x,y,z)$$

JOINT PMFs OF MULTIPLE R.V.s
More Than Two R.V.s

The <u>expected value rule</u>

$$E\left[g\left(X,Y,Z\right)\right]=\sum_x\sum_y\sum_z g\left(x,y,z\right)p_{X,Y,Z}\left(x,y,z\right)$$

If g is **linear**

$$E\left[g\left(X,Y,Z\right)\right]=E\left[aX+bY+cZ+d\right]=aE\left[X\right]+bE\left[Y\right]+cE\left[Z\right]+d$$

Generalization:

$$E\left[a_1X_1+a_2X_2+\cdots+a_nX_n\right]=a_1E\left[X_1\right]+a_2E\left[X_2\right]+\cdots+a_nE\left[X_n\right]$$

JOINT PMFs OF MULTIPLE R.V.s
More Than Two R.V.s
Let us return to Example 2.2. (Mean of the Binomial R.V.)

Experiment: independent coin tosses, with a 3/4 probability of head,
X : the number of heads obtained. E[X]=?

A binomial random variable with parameters n = 2 and p = 3/4. PMF

$$p_X(k) = \begin{cases} (1/4)^2, & \text{if } k = 0, \\ 2.(1/4)(3/4), & \text{if } k = 1, \\ (3/4)^2, & \text{if } k = 2. \end{cases}$$

$$E[X] = \sum_{k \in \{0,1,2\}} k p_X(k) = \sum_{k=0}^{2} k p_X(k) = 0 \times \frac{1}{16} + 1 \times \frac{6}{16} + 2 \times \frac{9}{16} = \frac{24}{16} = \frac{3}{2}$$

JOINT PMFs OF MULTIPLE R.V.s

Example 2.10 Mean of the Binomial R.V. (textbook) Probability class has 300 students and the probability that each student getting an A is 1/3. Grades of students are independent.E[*X*]=?, the number of students that get an A?

$$X_i = \begin{cases} 1, & \text{if the } i\text{th student gets an A,} \\ 0, & \text{ortherwise.} \end{cases}$$

$X_1, X_2, \ldots, X_n$ are Bernoulli random variables with mean $p = 1/3$. Since X is the number of "successes" in *n* independent trials,

$$X = X_1 + X_2 + \cdots + X_n$$

$$E[X] = E[X_1 + X_2 + \cdots + X_{300}] = \sum_{i=1}^{300} E[X_i] = 300 \times \frac{1}{3} = 100 \text{ (linearity)}$$

Generalization : $E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

JOINT PMFs OF MULTIPLE R.V.s

Example 2.11 (textbook). The Hat Problem.

Let

$$X_i = \begin{cases} 1, & \text{if the } i\text{th person selects his/her own hat,} \\ 0, & \text{ortherwise.} \end{cases}$$

$$P(X_i = 1) = 1/n, \quad P(X_i = 0) = 1 - 1/n, \text{ and } E[X_i] = 1 \times \frac{1}{n} + 0 \times \left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$E[X] = E[X_1 + X_2 + \cdots + X_n] = \sum_{i=1}^{n} E[X_i] = n \times \frac{1}{n} = 1$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.