CONDITIONING

Conditioning a Random Variable on an Event

$$p_{X|A}(x) = P(X = x \mid A) = \frac{P(\{X = x\} \cap A)}{P(A)}$$

Example 2.12 (textbook)

$$p_{X|A}(x) = P(X = x \mid \text{roll is even}) = \frac{P(X = x \text{ and } X \text{ is even})}{P(\text{roll is even})} = \begin{cases} 1/3 & \text{if } x = 2,4,6 \\ 0 & \text{otherwise} \end{cases}$$

- If $A_1, \ldots, A_n$ are disjoint events that form a partition of the sample space,

$$p_X(x) = \sum_{i=1}^{n} P(A_i)\, p_{X|A_i}(x)$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONING
Conditioning one Random Variable on Another

Conditional PMF $p_{X|Y}(x \mid y)$

$$p_{X|Y}(x \mid y) = P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

- For a fixed value of *y*, $p_{X|Y}(x \mid y)$ is a function of *x*.
- This function is a <u>legitimate PMF</u>:
  - nonnegative values to each *x*
  - normalization:

$$\sum_x p_{X|Y}(x \mid y) = \sum_x \frac{p_{X,Y}(x, y)}{p_Y(y)} = \frac{1}{p_Y(y)} \sum_x p_{X,Y}(x, y) = \frac{p_Y(y)}{p_Y(y)} = 1$$

CONDITIONING
Conditioning one Random Variable on Another

$p_{X|Y}(x|y)$ has the **same shape** as $p_{X,Y}(x,y)$, but at a different scale.

Sequential approach

$$p_{X,Y}(x,y) = p_Y(y)\, p_{X|Y}(x|y)$$

or equivalently

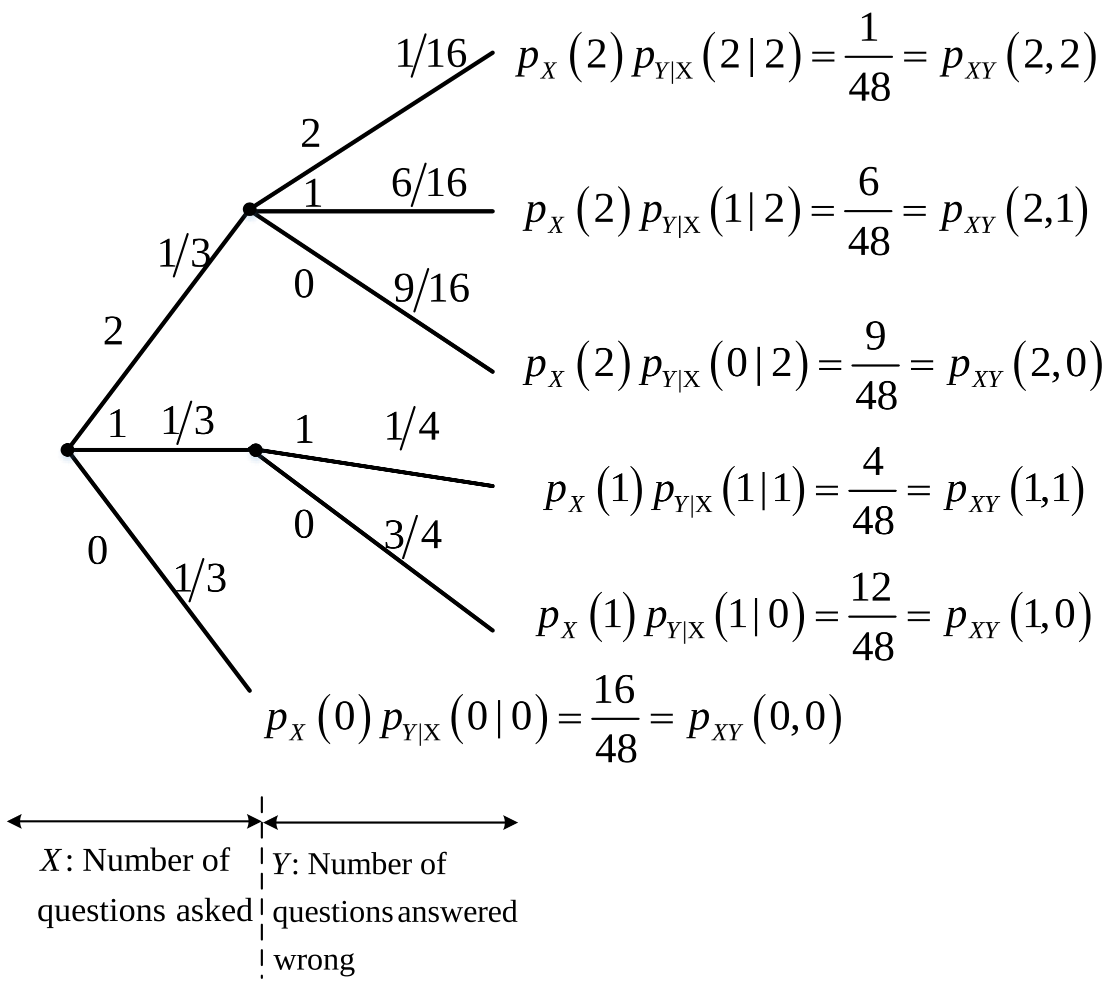
$$p_{X,Y}(x,y) = p_X(y)\, p_{\mathrm{Y|X}}(y|x)$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONING

Conditioning one Random Variable on Another

Example 2.14 (textbook)



$p_{XY}(x,y)$

$y \backslash x$	0	1	2
2	0	0	1/48
1	0	4/48	6/48
0	16/48	12/48	9/48

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

CONDITIONING
Conditioning one Random Variable on Another

The **conditional PMF** can be used to calculate the **marginal PMFs**.

$$p_X(x) = \sum_y p_{X,Y}(x,y) = \sum_y p_Y(y)\, p_{X|Y}(x \mid y)$$

divide-and-conquer approach for calculating marginal PMFs (identical to the **total probability theorem)**

CONDITIONING
Conditioning one Random Variable on Another
Example 2.15 (textbook)

$$p_{X|Y}\left(x\,|\,10^2\right)=\begin{cases}1/2 & \text{if } x=10^{-2}\\ 1/3 & \text{if } x=10^{-1}\\ 1/6 & \text{if } x=1\end{cases}\qquad p_{X|Y}\left(x\,|\,10^4\right)=\begin{cases}1/2 & \text{if } x=1\\ 1/3 & \text{if } x=10\\ 1/6 & \text{if } x=100\end{cases}$$

$$p_X(x)=\sum_y p_Y(y)\,p_{X|Y}(x\,|\,y)$$

We obtain

$$p_X\left(10^{-2}\right)=\frac{5}{6}\cdot\frac{1}{2}=\frac{5}{12},\qquad p_X\left(10^{-1}\right)=\frac{5}{6}\cdot\frac{1}{3}=\frac{5}{18},\ p_X(1)=\frac{5}{6}\cdot\frac{1}{6}+\frac{1}{6}\cdot\frac{1}{2}=\frac{8}{36}$$

$$p_X(10)=\frac{1}{6}\cdot\frac{1}{3}=\frac{1}{18},\qquad p_X(100)=\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{36}$$

INDEPENDENCE

Independence of Random Variables

X and *Y* are **independent** if

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y), \qquad \text{for all } x, y.$$

By using the equality $p_{X,Y}(x,y) = p_{X|Y}(x \mid y)\, p_Y(y)$, independence is equivalent to the condition

$$p_{X|Y}(x \mid y) = p_X(x), \quad \text{for all } y \text{ with } p_Y(y) > 0 \text{ and all } x$$

The value of *Y* does not provide any information for the value of *X*.

INDEPENDENCE
Independence of Random Variables

If X and Y are independent random variables, then

$$E[XY] = E[X]E[Y]$$

Verification:

$$\begin{aligned} E[XY] &= \sum_x \sum_y xy p_{X,Y}(x,y) \\ &= \sum_x \sum_y xy p_X(x) p_Y(y) \qquad \text{(by independence)} \\ &= \sum_x x p_X(x) \sum_y y p_Y(y) \\ &= E[X]E[Y] \end{aligned}$$

INDEPENDENCE

Independence of Random Variables

Let X and Y be independent random variables, $\text{var}(X+Y)=?$

$$\text{var}(X+Y)=\text{var}(\tilde{X}+\tilde{Y})=E\left[(\tilde{X}+\tilde{Y})^2\right]=E\left[\tilde{X}^2+2\tilde{X}\tilde{Y}+\tilde{Y}^2\right]$$
$$=E\left[\tilde{X}^2\right]+2E\left[\tilde{X}\tilde{Y}\right]+E\left[\tilde{Y}^2\right]$$
$$=E\left[\tilde{X}^2\right]+E\left[\tilde{Y}^2\right]$$
$$=\text{var}(\tilde{X})+\text{var}(\tilde{Y})$$
$$=\text{var}(X)+\text{var}(Y)$$

INDEPENDENCE
Independence of Random Variables

Alternatively,

$$\mathrm{var}(X+Y) = E\left[\left(X+Y-E[X+Y]\right)^2\right] = E\left[\left(X+Y-E[X]-E[Y]\right)^2\right]$$

$$= E\left[\left(X-E[X]\right)^2 - 2\left(X-E[X]\right)\left(Y-E[Y]\right) + \left(Y-E[Y]\right)^2\right]$$

$$= E\left[\left(X-E[X]\right)^2\right] + E\left[\left(Y-E[Y]\right)^2\right] - 2\underbrace{E\left[\left(X-E[X]\right)\left(Y-E[Y]\right)\right]}_{E\left[\left(X-E[X]\right)\right]E\left[\left(Y-E[Y]\right)\right]=0 \text{ (by independence)}}$$

$$= E\left[\left(X-E[X]\right)^2\right] + E\left[\left(Y-E[Y]\right)^2\right]$$

$$= \mathrm{var}(X) + \mathrm{var}(Y)$$

INDEPENDENCE
Variance of the Sum of Independent R.V.s

If $X_1, X_2, \ldots, X_n$ are **independent** random variables, then

$$\mathrm{var}(X_1 + X_2 + \cdots + X_n) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \cdots + \mathrm{var}(X_n)$$

- If $X_1, X_2, \ldots, X_n$ are **independent**

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$
$$\mathrm{var}(X_1 + X_2 + \cdots + X_n) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \cdots + \mathrm{var}(X_n)$$
$$\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$$

- If $X_1, X_2, \ldots, X_n$ are **not independent** r.v.s

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$