# GENERAL RANDOM VARIABLES

**Probability density function (*PDF*)** of *X*

$$P(X \in B) = \int_B f_X(x)\,dx$$

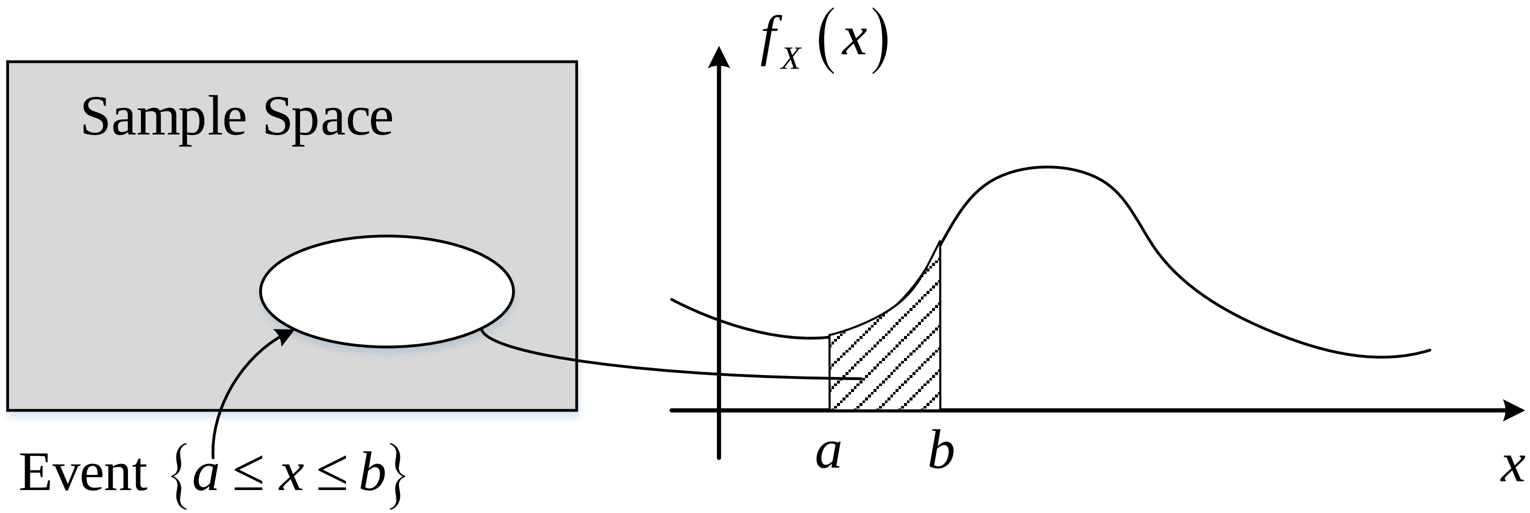
In particular, the probability that the value of *X* falls within an interval is

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

Note that this is the area under the graph of the PDF.

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

# CONTINUOUS RANDOM VARIABLES AND PDFS




Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

# CONTINUOUS RANDOM VARIABLES AND PDFS

For any single value $a$,

$$P(X=a)=\int_a^a f_X(x)\,dx=0$$

Therefore

$$P(a\le X\le b)=P(a<X<b)=P(a\le X<b)=P(a<X\le b)$$

legitimate PDF:

$f_X(x)\ge 0$ for every $x$ $\rightarrow$ Nonnegativity

$\int_{-\infty}^{\infty} f_X(x)\,dx=P(-\infty<x<\infty)=1$ $\rightarrow$ Normalization

↓

*Entire area under the PDF must be equal to* 1.

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# CONTINUOUS RANDOM VARIABLES AND PDFS

**Uniform** or **uniformly distributed** in the interval $\left[a,b\right]$.

PDF:

$$f_X\left(x\right)=\begin{cases}\dfrac{1}{b-a}, & \text{if } a\le x\le b\\ 0, & \text{otherwise}\end{cases}$$

$$1=\int_{-\infty}^{\infty} f_X\left(x\right)dx=\int_a^b c\,dx=c\left(b-a\right)\quad\Rightarrow\quad c=\frac{1}{b-a}$$

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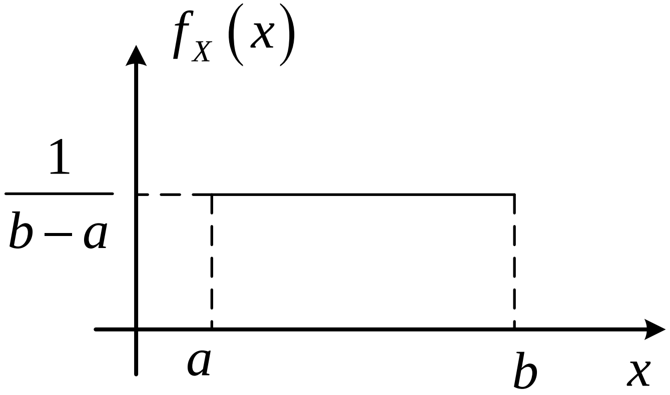
# CONTINUOUS RANDOM VARIABLES AND PDFS




The probability $P(X \in I)$ that $X$ takes value in a set $I$ is

$$P(X \in I) = \int_{[a,b]\cap I} f_X(x)dx = \int_{[a,b]\cap I} \frac{1}{b-a} dx = \frac{\text{length of } [a,b]\cap I}{\text{length of } [a,b]}$$

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# CONTINUOUS RANDOM VARIABLES AND PDFS

**Example 3.3 (textbook). A PDF can take arbitrarily large values.**
Consider a random variable *X* with PDF

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & \text{if } 0 < x \le 1, \\ 0 & \text{otherwise} \end{cases}$$

As *x* approaches zero $f_X(x)$ becomes infinitely large. However, $f_X(x)$ is a valid PDF. Since

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}\Big|_0^1 = 1$$

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# CONTINUOUS RANDOM VARIABLES AND PDFS

## Expectation

The **expected value** or **mean**

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$Y = g(X)$ is also a random variable.

The mean of $g(X)$ can be obtained by using the **expected value rule:**

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx$$

# CONTINUOUS RANDOM VARIABLES AND PDFS

## Expectation

- The ***n*****th moment** $E\left[X^n\right]$

- The **variance** of $X$ is defined

$$\operatorname{var}(X) = E\left[\left(X - E[X]\right)^2\right] = \int_{-\infty}^{\infty} \left(X - E[X]\right)^2 f_X(x)\,dx$$

- $0 \le \operatorname{var}(X) = E\left[X^2\right] - \left(E[X]\right)^2$

- If $Y = aX + b$, then

$$E[Y] = E[aX + b] = aE[X] + b \qquad \operatorname{var}(Y) = a^2 \operatorname{var}(X)$$

# CONTINUOUS RANDOM VARIABLES AND PDFS

## Expectation

**Example 3.4 (textbook). Mean and variance of the Uniform r.v.**
Consider a uniform random variable *X*, interval $[a,b]$.

Find the mean and variance of *X*.

$$E[X]=\int_{-\infty}^{\infty} x f_X(x)\,dx=\int_a^b x\frac{1}{b-a}\,dx=\frac{1}{b-a}\frac{1}{2}x^2\Big|_a^b$$

$$=\frac{b^2-a^2}{b-a}\frac{1}{2}=\frac{a+b}{2}$$

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# CONTINUOUS RANDOM VARIABLES AND PDFS
## Expectation

### Example 3.4. (Continued)

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\frac{1}{3}x^3\Big|_a^b$$

$$= \frac{b^3-a^3}{b-a}\frac{1}{3} = \frac{a^2+ab+b^2}{3}$$

Therefore, the variance is obtained as

$$\mathrm{var}(X) = E\left[X^2\right] - \left(E[X]\right)^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

# CONTINUOUS RANDOM VARIABLES AND PDFS
## Exponential Random Variable

PDF of an **exponential** random variable is given as

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda$ is a positive parameter.

This is a legitimate PDF since

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{\infty} = 1$$

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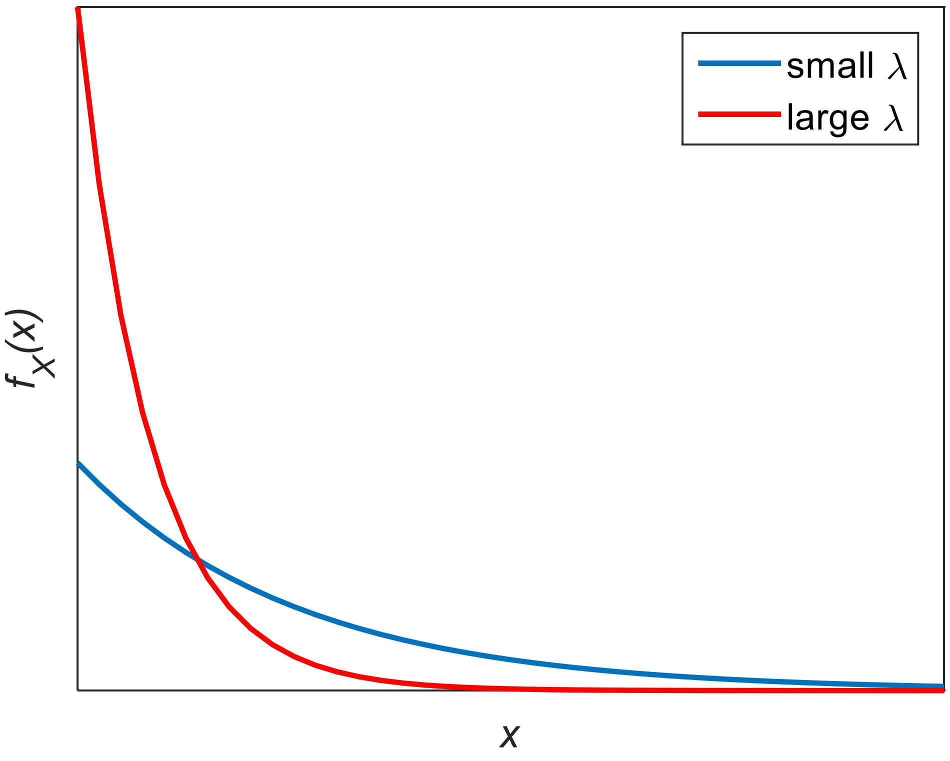
# CONTINUOUS RANDOM VARIABLES AND PDFS

## Exponential Random Variable

The PDF of an exponential random variable.




- Note that $\lambda$ characterizes the PDF.

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# CONTINUOUS RANDOM VARIABLES AND PDFS

## Exponential Random Variable

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{0}^{\infty} x\lambda e^{-\lambda x}\,dx$$

$$= \left(-x\lambda e^{-\lambda x}\right)\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx = 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^{\infty} = \frac{1}{\lambda}$$

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{0}^{\infty} x^2\lambda e^{-\lambda x}\,dx$$

$$= \left(-x^2\lambda e^{-\lambda x}\right)\Big|_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x}\,dx = 0 + \frac{2}{\lambda}E[X] = \frac{2}{\lambda^2}$$

Variance of the exponential random variable $X$ is

$$\operatorname{var}(X) = E\left[X^2\right] - E\left([X]\right)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$