CUMULATIVE DISTRIBUTION FUNCTIONS

Cumulative Distribution Function (CDF)

$$F_X(x) = P(X \le x) = \begin{cases} \sum_{k \le x} p_X(k) & X \text{ is discrete,} \\ \int_{-\infty}^{x} f_X(t)\,dt & X \text{ is continuous} \end{cases}$$

Example: Let X be a uniform random variable in the interval $[a,b]$. Find the CDF of X.

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b, \\ 0 & \text{otherwise} \end{cases}$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CUMULATIVE DISTRIBUTION FUNCTIONS

Example-continued

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(z)\,dz$$

$$x < a \quad \Rightarrow \quad F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz = 0$$

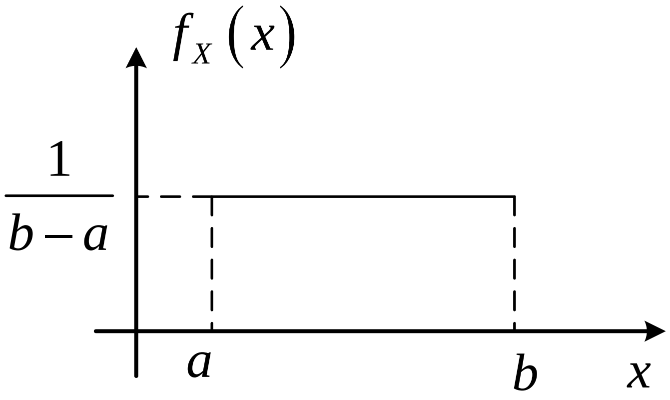
$$a \le x < b \Rightarrow \quad F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz$$

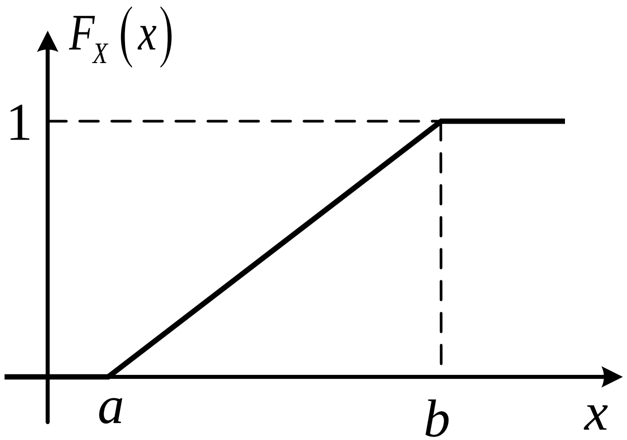
$$= \frac{1}{b-a}\int_{a}^{x} dz = \frac{1}{b-a}\, z\Big|_a^x = \frac{x-a}{b-a}$$

$$x \ge b \quad \Rightarrow \quad F_X(x) = \int_{-\infty}^{b} f_X(z)\,dz$$

$$= \frac{1}{b-a}\int_{a}^{b} dz = \frac{1}{b-a}\, z\Big|_a^b = \frac{b-a}{b-a} = 1$$

CUMULATIVE DISTRIBUTION FUNCTIONS
Example-continued





Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CUMULATIVE DISTRIBUTION FUNCTIONS
Properties

- if $x \le y$, then $F_X(x) \le F_X(y)$
- $0 \le F_X(x) \le 1$
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F_X(x) = 0$
- $P(X > x) = 1 - F_X(x)$

NORMAL (GAUSSIAN) RANDOM VARIABLES

A Gaussian random variable can be employed for modeling

- Thermal noise
- Measurement error

Gaussian PDF

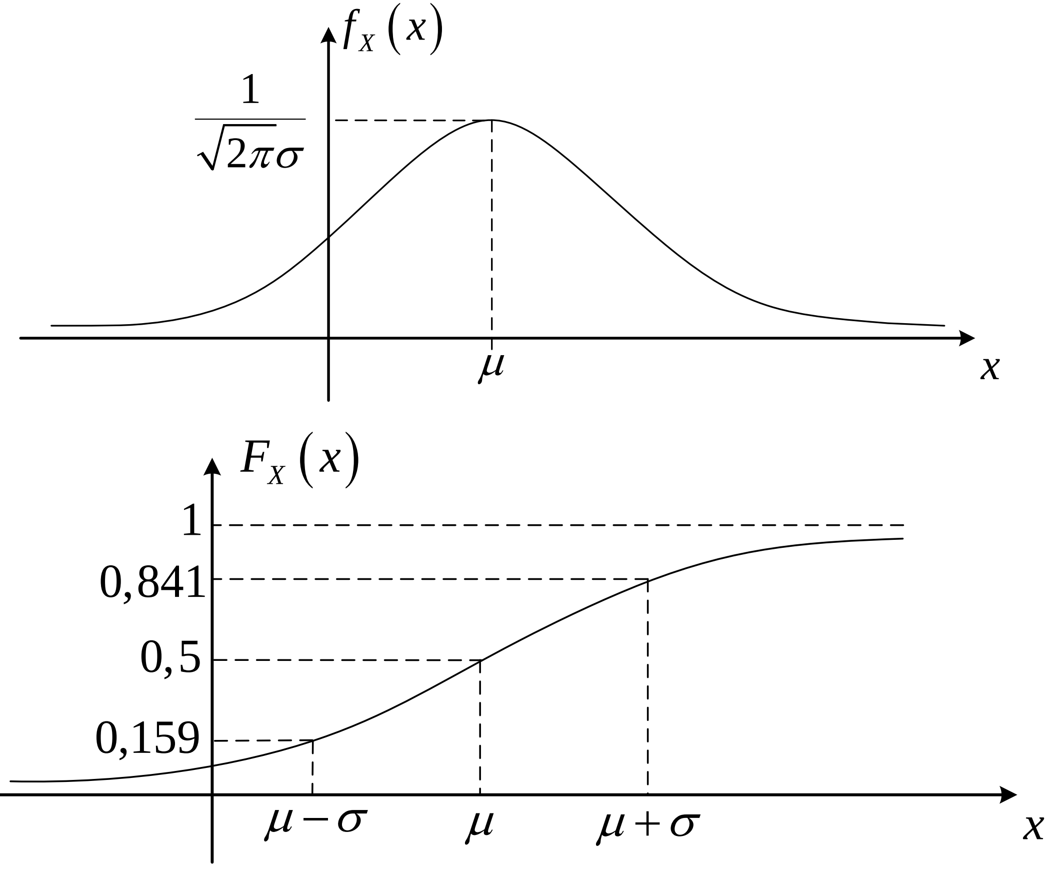
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty$$

where μ is **mean** and σ is the **standard deviation**.

$X \sim N\left(\mu, \sigma^2\right) \rightarrow$ ***X is Gaussian distributed with mean*** μ ***and variance*** σ^2

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

GAUSSIAN RANDOM VARIABLES



Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

GAUSSIAN RANDOM VARIABLES

$$E[X] = \mu$$
$$\text{var}(X) = \sigma^2$$

Normalization property the Gaussian PDF :

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-(x-\mu)^2 / 2\sigma^2\right] dx = 1$$

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(z-\mu)^2/2\sigma^2} dz$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

GAUSSIAN RANDOM VARIABLES

Normality is preserved by linear transformation

$$Y = aX + b$$

Y is also a normal random variable with mean and variance:

$$E[Y] = E[aX + b] = aE[X] + b$$

$$\text{var}(Y) = \text{var}(aX + b) = a^2 \text{var}(X)$$

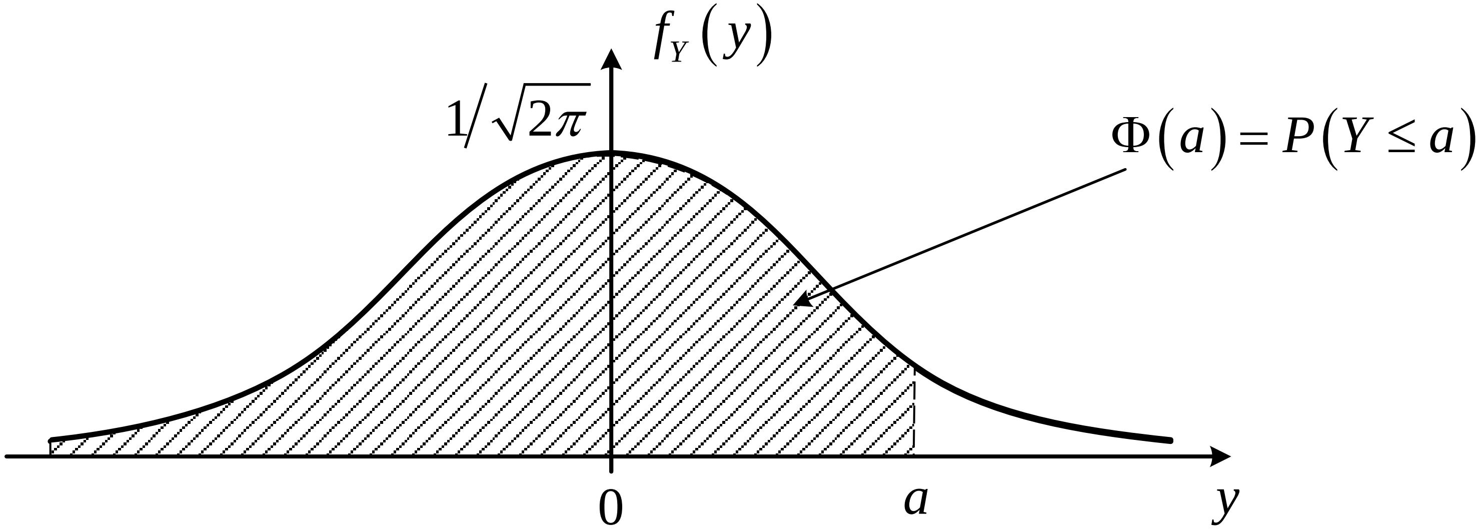
GAUSSIAN RANDOM VARIABLES

The Standard Normal Random Variable

<u>zero mean</u> and <u>unit variance</u>

- **CDF of standard normal random variable is represented by Φ**

$$\Phi(y) \triangleq F_Y(y) = P(Y \le y) = P(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-t^2/2\right) dt$$



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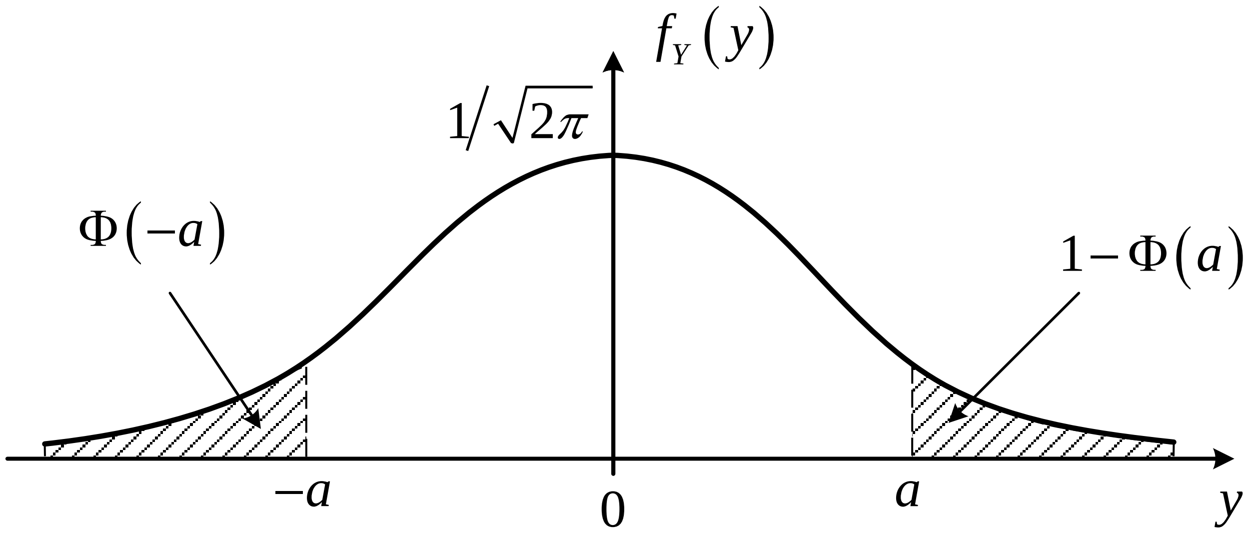
GAUSSIAN RANDOM VARIABLES
The Standard Normal Random Variable

$$\Phi(-y) = P(Y \le -y) = P(Y > y) = 1 - P(Y \le y) = 1 - \Phi(y)$$



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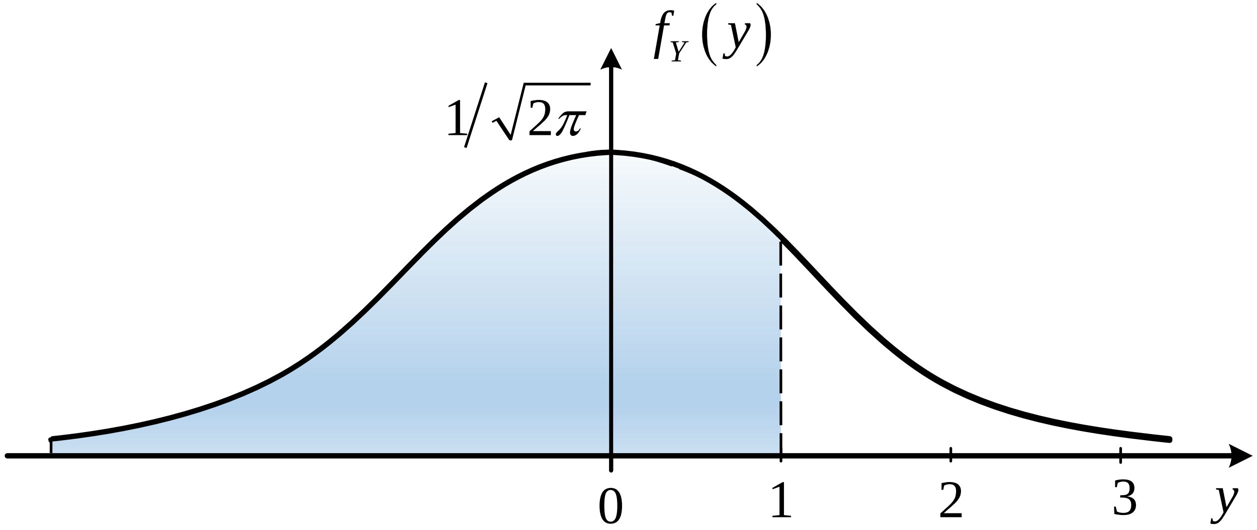
GAUSSIAN RANDOM VARIABLES
The Standard Normal Random Variable

Example: Consider the standard normal random variable X.

$P\left(X \le \sigma\right) = ?, P\left(X \le 2\sigma\right) = ?, P\left(X \le 3\sigma\right) = ? \, P\left(\left|X\right| \le \sigma\right) = ?, P\left(\left|X\right| \le 2\sigma\right) = ?, P\left(\left|X\right| \le 3\sigma\right) = ?$

$$P\left(X \le \sigma\right) = F_X\left(1\right) = \Phi\left(1\right) = 0.84134$$



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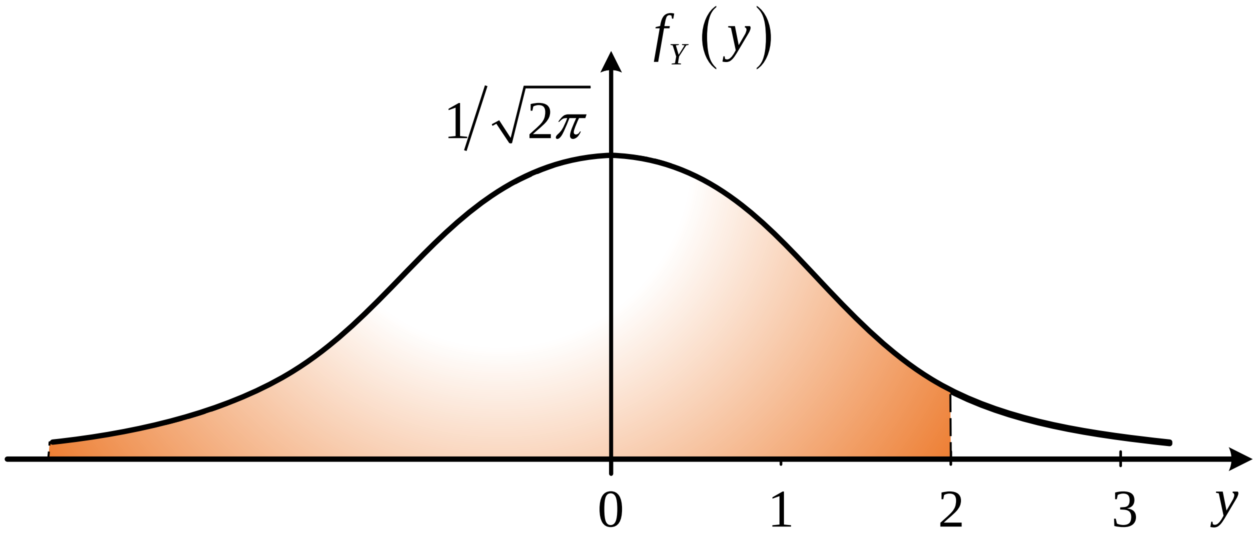
GAUSSIAN RANDOM VARIABLES
The Standard Normal Random Variable
Example (Continued)

$$P(X \le 2\sigma) = F_X(2) = \Phi(2) = 0.97725$$



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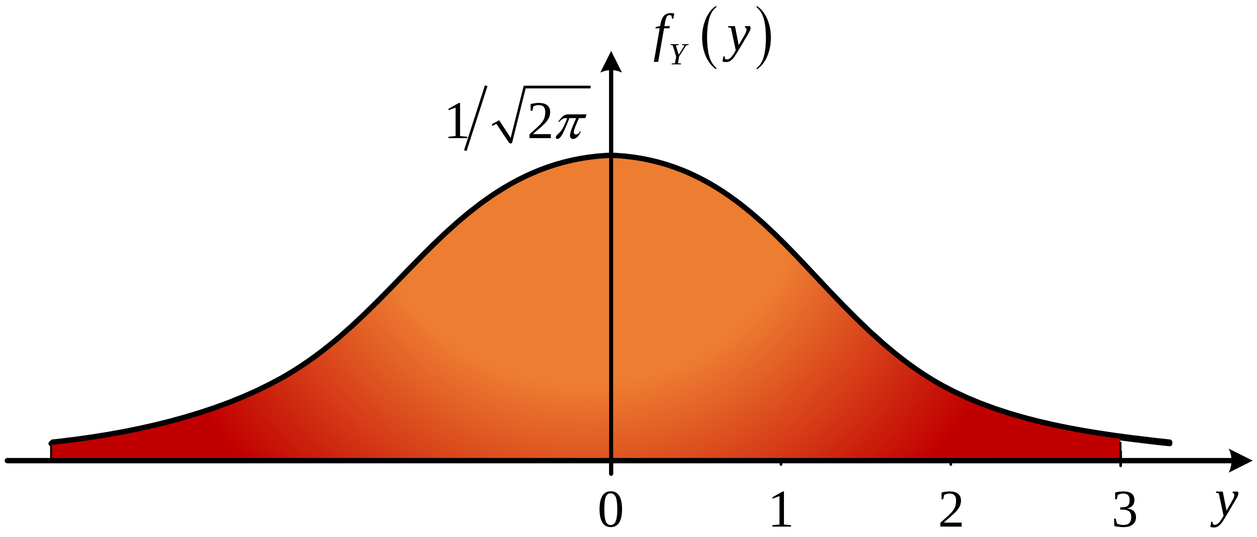
GAUSSIAN RANDOM VARIABLES
The Standard Normal Random Variable
Example (Continued)

$$P\left(X \leq 3\sigma\right) = F_X\left(3\right) = \Phi\left(3\right) = 0.99865$$



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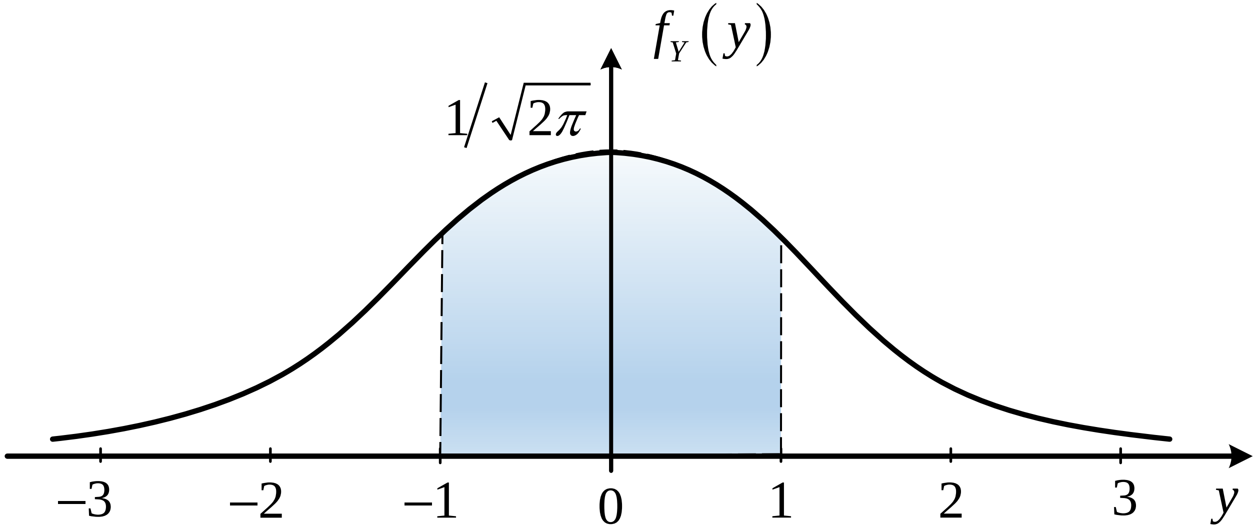
GAUSSIAN RANDOM VARIABLES

The Standard Normal Random Variable

Example (Continued)

$$P\left(\left|X\right| \le 1\sigma\right) = P\left(-\sigma \le X \le \sigma\right) = F_X\left(1\right) - F_X\left(-1\right)$$

$$= \Phi\left(1\right) - \underbrace{\Phi\left(-1\right)}_{1-\Phi(1)} = 2\Phi\left(1\right) - 1 = 2 \times 0.84134 - 1 = 0.68262$$



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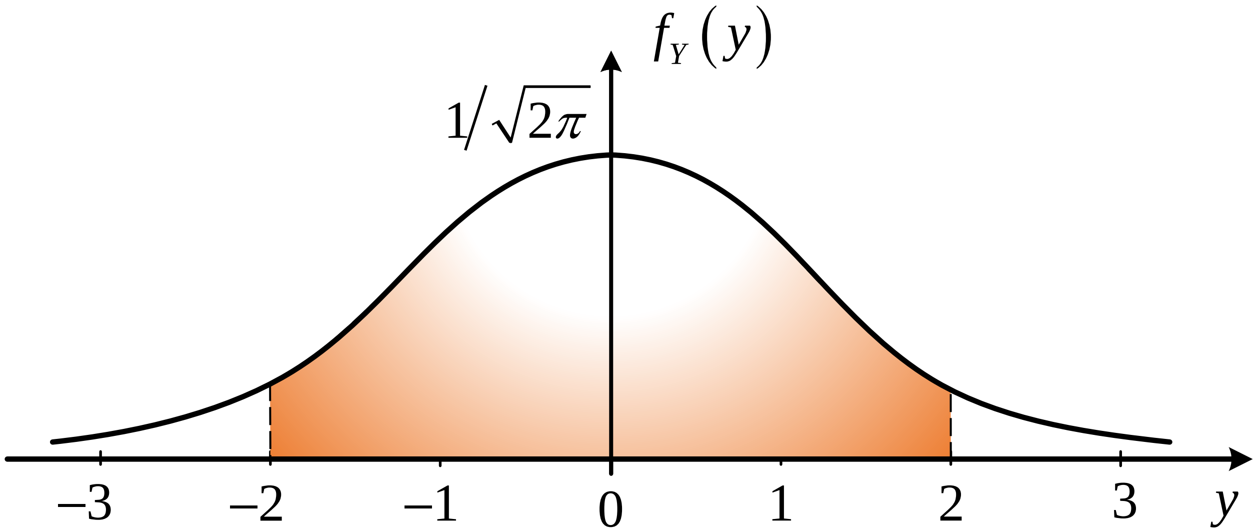
GAUSSIAN RANDOM VARIABLES
The Standard Normal Random Variable
Example (Continued)

$$P\left(|X| \le 2\sigma\right) = P\left(-2\sigma \le X \le 2\sigma\right) = F_X\left(2\right) - F_X\left(-2\right)$$

$$= \Phi\left(2\right) - \underbrace{\Phi\left(-2\right)}_{1-\Phi(2)} = 2\Phi\left(2\right) - 1 = 2 \times 0.97725 - 1 = 0.9545$$



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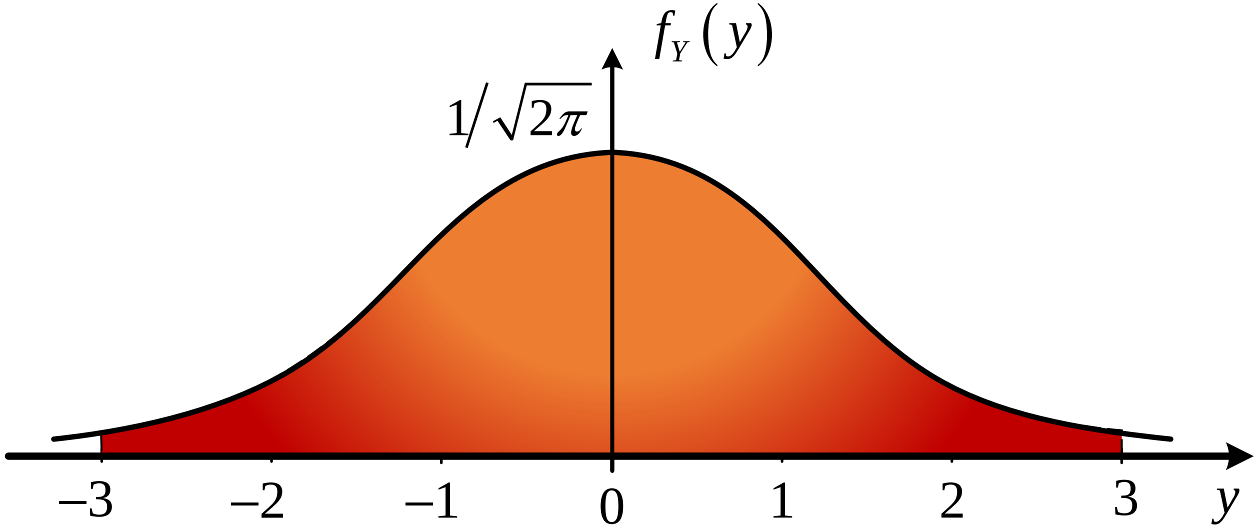
GAUSSIAN RANDOM VARIABLES
The Standard Normal Random Variable
Example (Continued)

$$P\left(\left|X\right| \le 3\sigma\right) = P\left(-3\sigma \le X \le 3\sigma\right) = F_X\left(3\right) - F_X\left(-3\right)$$

$$= \Phi\left(3\right) - \underbrace{\Phi\left(-3\right)}_{1-\Phi\left(3\right)} = 2\Phi\left(3\right) - 1 = 2 \times 0.99865 - 1 = 0.9973$$



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GAUSSIAN RANDOM VARIABLES

How to Use the Table for a <u>Nonstandard</u> Gaussian Random Variable

Let X be a normal a random variable with mean μ and variance σ^2.We can "**<u>standardize</u>**" X by defining a new random variable Y given by

$$Y = \frac{X - \mu}{\sigma}$$

Standardization

Since Y is a <u>linear transformation</u> of X then Y would be normal random variable with mean 0 and variance 1.

$$E[Y] = E\left[\frac{X - \mu}{\sigma}\right] = \frac{E[X] - \mu}{\sigma} = 0, \qquad \mathrm{var}(Y) = \frac{\mathrm{var}(X)}{\sigma^2} = 1$$

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