JOINT PDFs OF MULTIPLE RANDOM VARIABLES

Joint PDF $f_{X,Y}$ satisfies

$$P\left(\left(X,Y\right)\in B\right)=\iint_{(x,y)\in B} f_{X,Y}\left(x,y\right)dxdy$$

In particular case where *B* is a rectangle: $B=\left\{\left(x,y\right)\mid a\le x\le b, c\le y\le d\right\}$, then

$$P\left(a\le X\le b, c\le Y\le d\right)=\int_c^d\int_a^b f_{X,Y}\left(x,y\right)dxdy$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}\left(x,y\right)dxdy=1 \quad \text{(normalization)}$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

JOINT PDFs OF MULTIPLE RANDOM VARIABLES

We see that the **marginal PDF** $f_X(x)$ of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

JOINT PDFs OF MULTIPLE RANDOM VARIABLES

Example 3.9 (textbook) Romeo and Juliet

$$f_{X,Y}(x,y)=\begin{cases} c, & \text{if } 0\le x\le 1 \text{ and } 0\le y\le 1, \\ 0, & \text{otherwise} \end{cases}$$

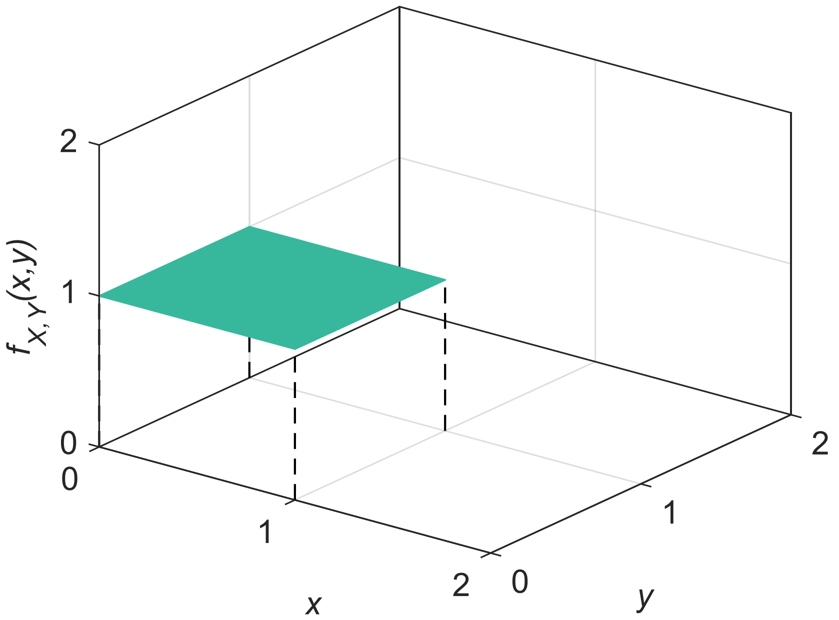
where c is a constant. Normalization property

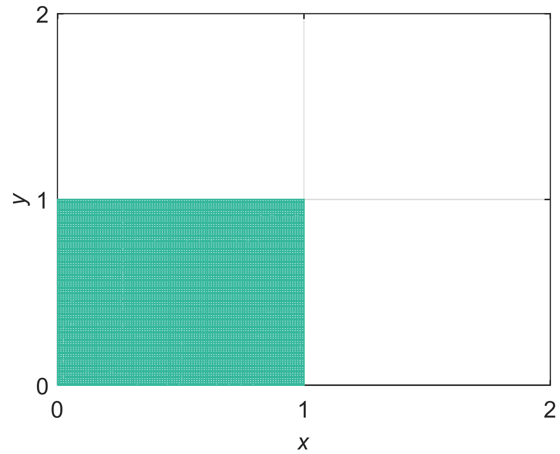
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dxdy=\int_{0}^{1}\int_{0}^{1} c\,dxdy=1 \quad\Rightarrow\quad c=1$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

JOINT PDFs OF MULTIPLE RANDOM VARIABLES

Example 3.9. (Continued)





The joint PDF in three-dimensional space

Top view of the joint PDF

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

JOINT PDFs OF MULTIPLE RANDOM VARIABLES

Uniform joint PDF on S

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{area of } S}, & \text{if } (x,y) \in S, \\ 0, & \text{otherwise} \end{cases}$$

For any set $A \subset S$, the probability that (X,Y) lies in A is

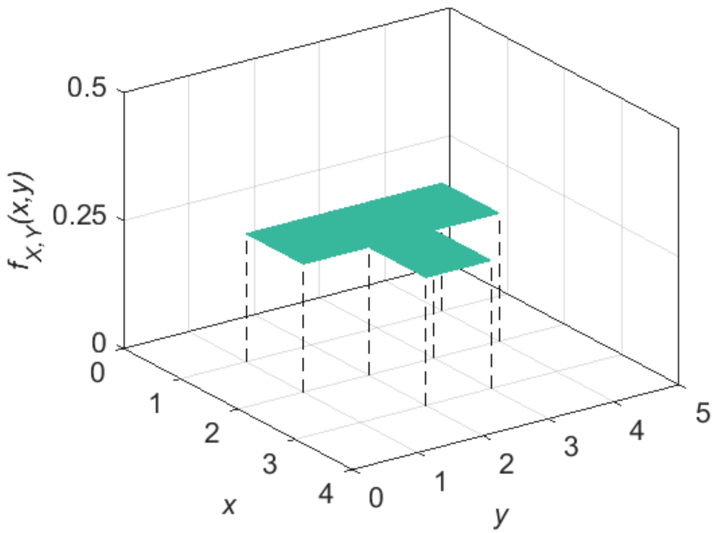
$$P\big((X,Y) \in A\big) = \iint_{(x,y)\in A} f_{X,Y}(\mathrm{x},y)\,dxdy = \frac{1}{\text{area of } S} \iint_{(x,y)\in A} dxdy = \frac{\text{area of } A}{\text{area of } S}$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

JOINT PDFs OF MULTIPLE RANDOM VARIABLES

Example 3.10. (textbook)

Joint PDF of X and Y is constant c on the set S. Find the marginal PDFs.



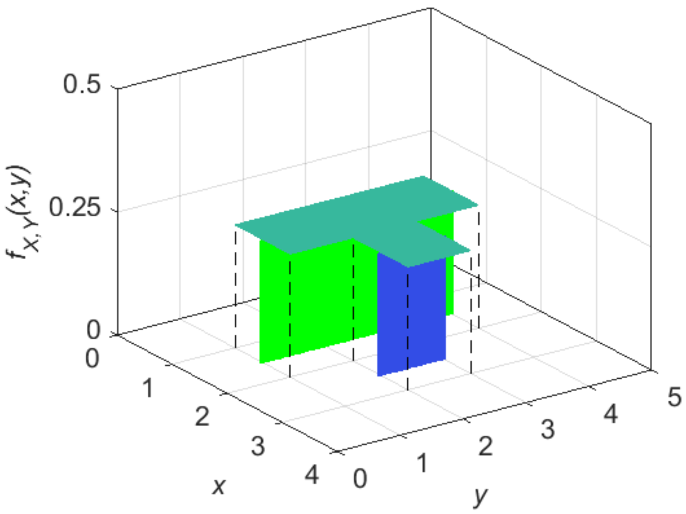
Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

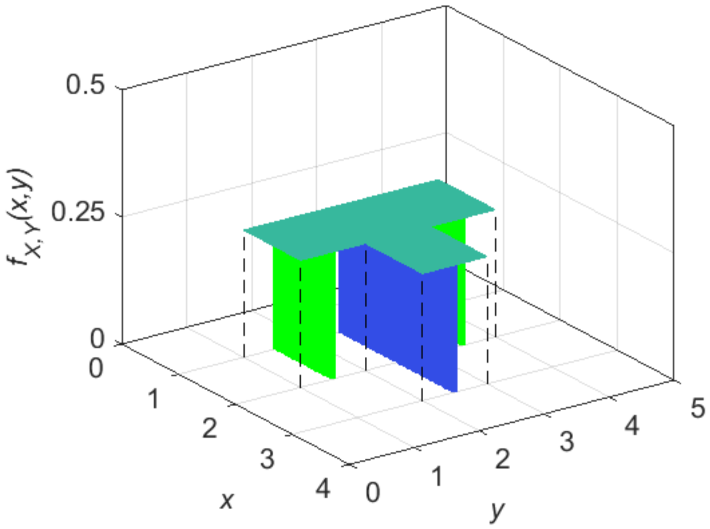
JOINT PDFs OF MULTIPLE RANDOM VARIABLES
Example 3.10. (Continued)

The area of the set S is equal to 4, therefore,

$$f_{X,Y}(x,y) = c = \frac{1}{4}, \qquad\qquad (x,y) \in S$$

To find the marginal PDFs, joint PDF is integrated over the vertical line.





Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

JOINT PDFs OF MULTIPLE RANDOM VARIABLES
Joint CDFs

Joint CDF is given as

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

If we know joint PDF $f_{X,Y}$, then

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds$$

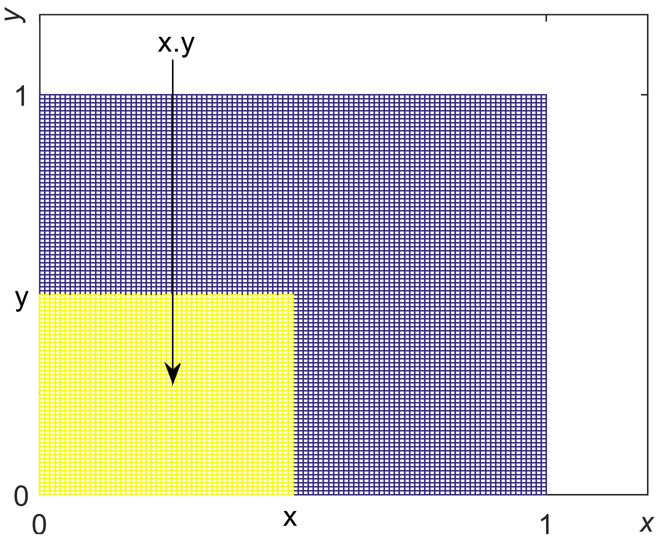
Conversely, the PDF can be obtained from the CDF:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

JOINT PDFs OF MULTIPLE RANDOM VARIABLES
Joint CDFs

Example 3.12. *X* and *Y* have a uniform PDF on the unit square. Find the joint CDF.

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)\,dtds = \int_{0}^{x}\int_{0}^{y} 1\,dtds = xy, \quad \text{for } 0 \le x, y \le 1$$



$$F_{X,Y}(x,y) = \begin{cases} 0, & \text{for } x, y \le 0 \\ xy, & \text{for } 0 < x, y \le 1 \\ 1, & \text{for } 1 < x, y \end{cases}$$

Verification

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = \frac{\partial^2 xy}{\partial x \partial y} = 1 = f_{X,Y}(x,y)$$

JOINT PDFs OF MULTIPLE RANDOM VARIABLES
Expectation

The **expected value rule**

$$E\left[g\left(X,Y\right)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(x,y\right) f_{X,Y}\left(x,y\right)dxdy$$

Special case, for any scalars *a*, *b*, and *c*, we have

$$E\left[aX+bY+c\right]=aE\left[X\right]+bE\left[Y\right]+c$$