CONDITIONING

CONDITIONING

Conditioning One Random Variable on Another

Conditional PDF of *X* given that $Y = y$, is defined by

$$f_{X|Y}\left(x \mid y\right) = \frac{f_{X,Y}\left(x,y\right)}{f_Y\left(y\right)}$$

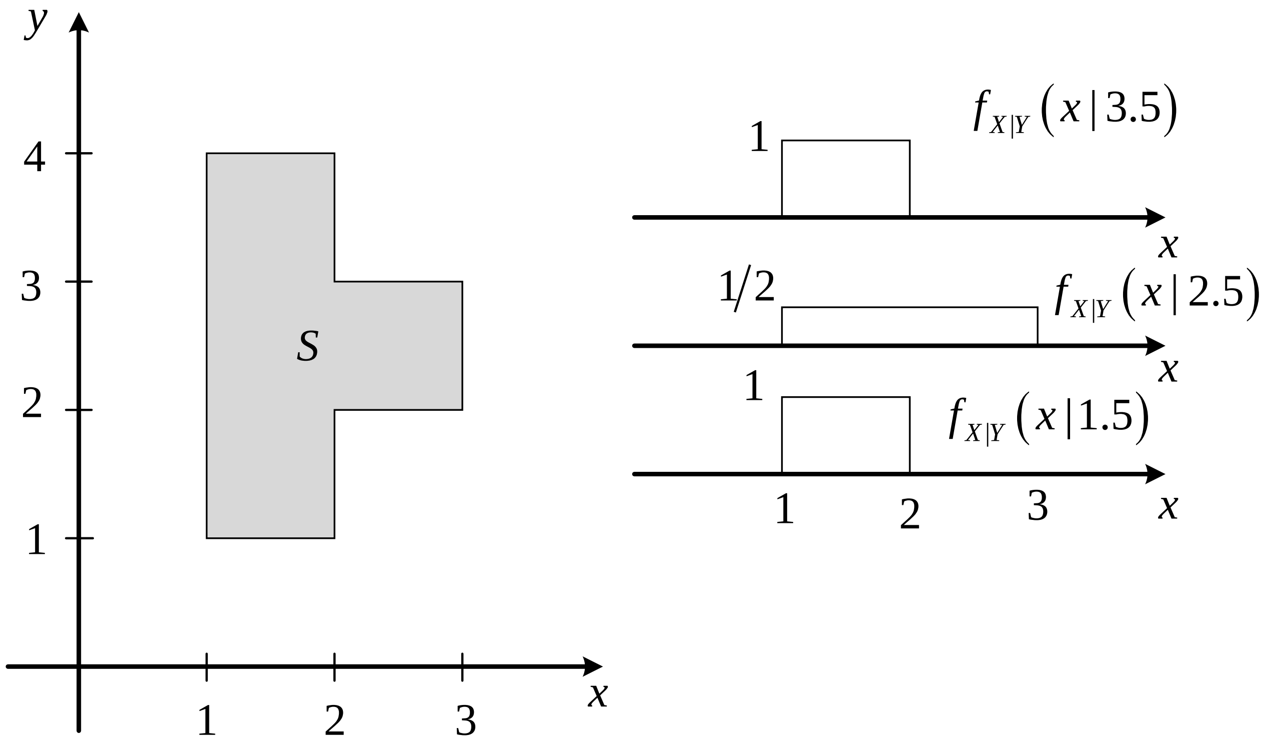
y as a fixed number and $f_{X|Y}\left(x \mid y\right)$ is as a function of the single variable *x*.

the conditional PDF $f_{X|Y}\left(x \mid y\right)$ has the same shape as the joint PDF $f_{X,Y}\left(x,y\right)$

Normalization $$\int_{-\infty}^{\infty} f_{X|Y}\left(x \mid y\right) dx = 1$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONING
Conditioning One Random Variable on Another



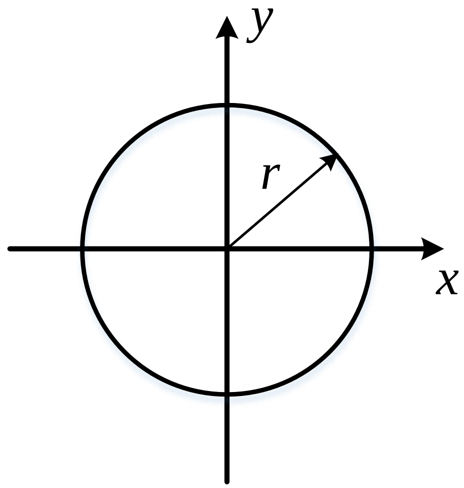
Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

CONDITIONING
Conditioning One Random Variable on Another

Example. 3.15 (textbook) Circular Uniform PDF.

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{area of the circle}}, & \text{if } (x,y) \text{ in the circle,} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1}{\pi r^2}, & \text{if } x^2 + y^2 \le r^2, \\ 0, & \text{otherwise} \end{cases}$$



Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONING
Conditioning One Random Variable on Another
Example 3.15. (Continued)

For $|y| > r$, $f_Y(y) = 0$. For $|y| \le r$, it is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \frac{1}{\pi r^2} \int_{x^2+y^2 \le r^2} dx = \frac{1}{\pi r^2} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx = \frac{2}{\pi r^2}\sqrt{r^2-y^2},\ |y| \le r$$

Note that the marginal PDF f_Y is not uniform.

The conditional PDF is,

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/\pi r^2}{\frac{2}{\pi r^2}\sqrt{r^2-y^2}} = \frac{1}{2\sqrt{r^2-y^2}}, \qquad \text{if } x^2+y^2 \le r^2$$

Thus, for a fixed value of y, $f_{X|Y}$ is uniform.

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONING
Conditioning One Random Variable on Another

For more than two random variables:

$$f_{X,Y|Z}(x,y\,|\,z)=\frac{f_{X,Y,Z}(x,y,z)}{f_Z(z)},\ \ if\ \ f_Z(z)>0$$

$$f_{X|Y,Z}(x\,|\,\mathrm{y},z)=\frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)},\ \ if\ \ f_{Y,Z}(y,z)>0$$

Analog of the multiplication rule:

$$f_{X,Y,Z}(x,y,z)=f_{X|\mathrm{Y},Z}(x\,|\,\mathrm{y},z)\,f_{\mathrm{Y}|Z}(y\,|\,z)\,f_Z(z)$$

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CONDITIONING
Conditional Expectation

- **Definition:**

$$E[X \mid A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\,dx$$

The conditional expectation of X given $Y = y$ is defined by

$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\,dx$$

- **The expected value rule:** For a function $g(X)$, we have

$$E[g(X) \mid A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\,dx \quad \text{and}$$

$$E[g(X) \mid Y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y)\,dx$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONING
Independence

Random variables X and Y are **independent** if their joint PDF can be written as the product of the marginal PDFs:

$$f_{XY}(x, y) = f_X(x)\, f_Y(y) \text{ for all } x, y.$$

By using the formula $f_{XY}(x, y) = f_{X|Y}(x \mid y)\, f_Y(y)$, independence means that

$$f_{X|Y}(x \mid y) = f_X(x) \text{ for all } y \text{ with } f_Y(y) > 0 \text{ and all } x.$$

Symmetrically,

$$f_{Y|X}(y \mid x) = f_Y(y) \text{ for all } x \text{ with } f_X(x) > 0 \text{ and all } y.$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONING
Independence
Independence implies that

$$F_{XY}(x,y)=P(X\le x, Y\le y)=P(X\le x)P(Y\le y)=F_X(x)F_Y(y)$$

If X and Y are independent, then

$$E\left[g(X)h(Y)\right]=E\left[g(X)\right]E\left[h(Y)\right]$$

where g and h are real-valued functions.

For more than two random variables:

$$f_{XYZ}(x,y,z)=f_X(x)\,f_Y(y)\,f_Z(z)\quad \text{for all } x, y, z.$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

CONDITIONING
Independence
Summary

- X and Y are **independent** if

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

- If X and Y are **independent**, then

$$E[XY] = E[X]E[Y]$$

Furthermore, for any functions g and h, the random variables $g(X)$ and $h(Y)$ are independent, and therefore

$$E\left[g(X)h(Y)\right] = E\left[g(X)\right]E\left[h(Y)\right]$$

- If X and Y are **independent**, then

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$$

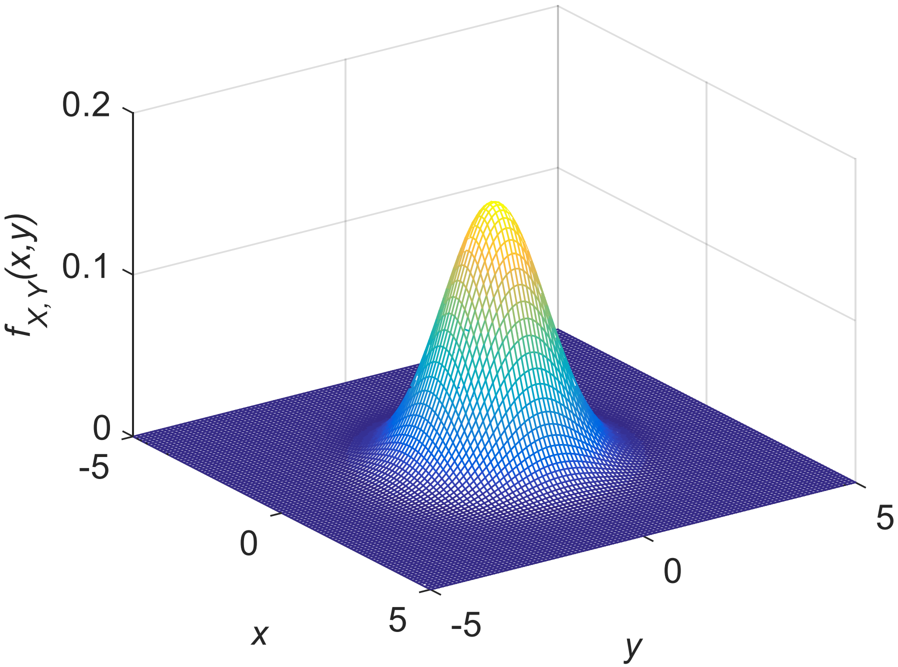
CONDITIONING
Independence

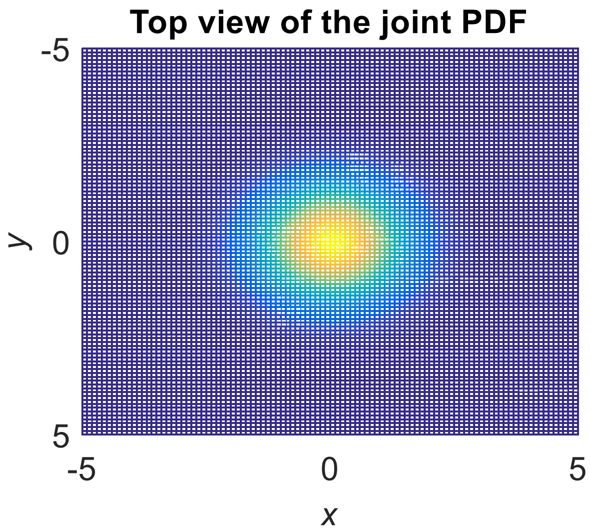
Example 3.18 (textbook). Independent Normal Random Variables

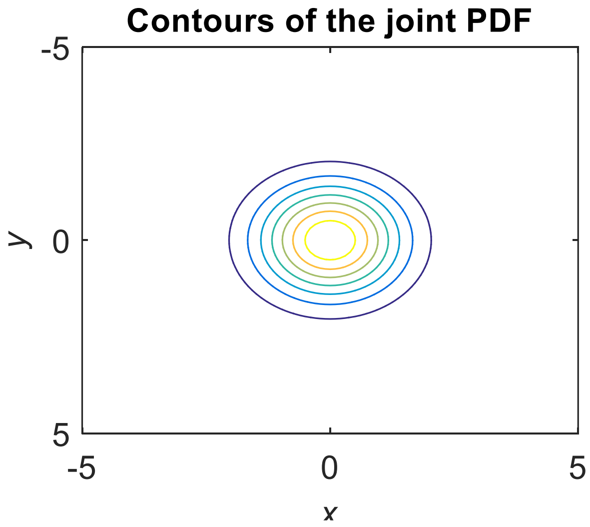
Let X and Y independent random variables. Their joint PDF is given as

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_x\sigma_y}\exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^{\ 2}} - \frac{(y-\mu_y)^2}{2\sigma_y^{\ 2}}\right)$$

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, "Introduction to Probability", 2nd Ed., Athena Science 2008.

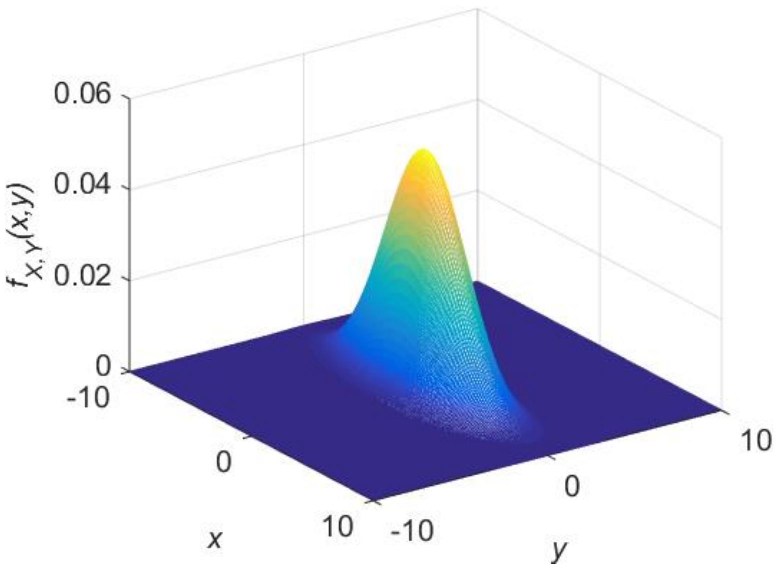


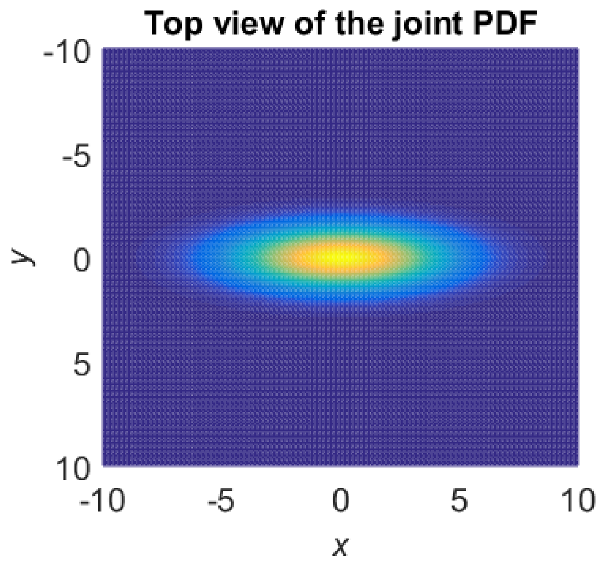


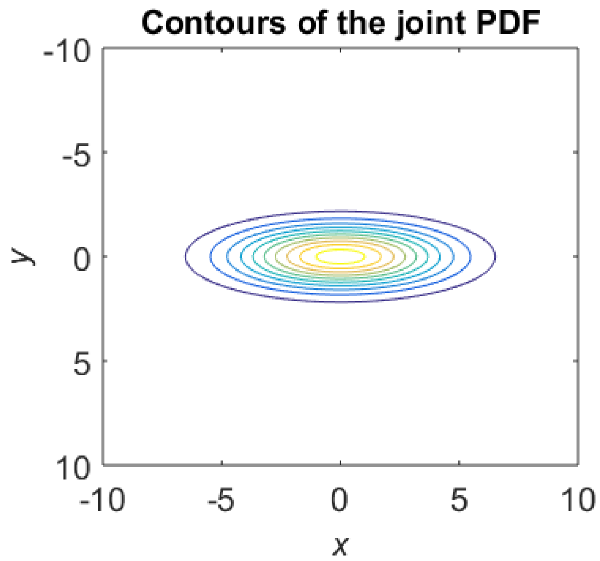


The joint PDF of two independent Normal r.v.s in three-dimensional space ($\sigma_X = 1$, $\sigma_Y = 1$)

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.







The joint PDF of two independent Normal r.v.s in three-dimensional space ($\sigma_X = 3,\ \sigma_Y = 1$)

Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.