10. LINEAR DIFFERENTIAL SYSTEMS

DIFFERENTIAL OPERATORS AND THE ELIMINATION METHOD FOR SYSTEMS

The notation $y'(t) = \frac{dy}{dt} = \frac{d}{dt}y$ was devised to suggest that the derivative of a function y is the result of *operating* on the function y with the differentiation operator $\frac{d}{dt}$. Indeed, second derivatives are formed by iterating the operation: $y''(t) = \frac{d^2y}{dt^2} = \frac{d}{dt}\frac{d}{dt}y$. Commonly, the symbol D is used instead of $\frac{d}{dt}$, and the second-order differential equation

$$y'' + 4y' + 3y = 0$$

is represented† by

$$D^2y + 4Dy + 3y = (D^2 + 4D + 3)[y] = 0 .$$

So, we have implicitly adopted the convention that the operator "product," D times D, is interpreted as the *composition* of D with itself, when it operates on functions: D^2y means $D(D[y])$; i.e., the second derivative. Similarly, the product $(D+3)(D+1)$ operates on a function via

$$\begin{aligned}(D+3)(D+1)[y] &= (D+3)\big[(D+1)[y]\big] = (D+3)[y'+y] \\ &= D[y'+y] + 3[y'+y] \\ &= (y''+y') + (3y'+3y) = y''+4y'+3y = (D^2+4D+3)[y]\ .\end{aligned}$$

Thus, $(D+3)(D+1)$ is the same operator as D^2+4D+3; when they are applied to twice-differentiable functions, the results are identical.

Example 1 Show that the operator $(D+1)(D+3)$ is also the same as D^2+4D+3.

Solution For any twice-differentiable function $y(t)$, we have

$$\begin{aligned}(D+1)(D+3)[y] &= (D+1)\big[(D+3)[y]\big] = (D+1)[y'+3y] \\ &= D[y'+3y] + 1[y'+3y] = (y''+3y') + (y'+3y) \\ &= y''+4y'+3y = (D^2+4D+3)[y]\ .\end{aligned}$$

Hence, $(D+1)(D+3) = D^2+4D+3$. ◆

Since $(D+1)(D+3) = (D+3)(D+1) = D^2 + 4D + 3$, it is tempting to generalize and propose that one can treat expressions like $aD^2 + bD + c$ as if they were ordinary polynomials in D. This is true, as long as we restrict the coefficients a, b, c to be *constants*. The following example, which has *variable* coefficients, is instructive.

Example 2 Show that $(D+3t)D$ is *not* the same as $D(D+3t)$.

Solution With $y(t)$ as before,

$$(D+3t)D[y] = (D+3t)[y'] = y'' + 3ty' \quad ;$$

$$D(D+3t)[y] = D[y' + 3ty] = y'' + 3y + 3ty' \quad .$$

They are not the same! ◆

This means that the familiar elimination method, used for solving *algebraic systems* like

$$
\begin{aligned}
3x - 2y + z &= 4 \ , \\
x + y - z &= 0 \ , \\
2x - y + 3z &= 6 \ ,
\end{aligned}
$$

can be adapted to solve *any system of linear differential equations with constant coefficients*.

Our goal in this section is to formalize this **elimination method** so that we can tackle more general linear constant coefficient systems.

We first demonstrate how the method applies to a linear system of two first-order differential equations of the form

$$a_1x'(t) + a_2x(t) + a_3y'(t) + a_4y(t) = f_1(t) ,$$
$$a_5x'(t) + a_6x(t) + a_7y'(t) + a_8y(t) = f_2(t) ,$$

where $a_1, a_2, \ldots, a_8$ are constants and $x(t), y(t)$ is the function pair to be determined. In operator notation this becomes

$$(a_1D + a_2)[x] + (a_3D + a_4)[y] = f_1 ,$$
$$(a_5D + a_6)[x] + (a_7D + a_8)[y] = f_2 .$$

Example 3 Solve the system

$$\text{(1)} \qquad \begin{aligned} x'(t) &= 3x(t) - 4y(t) + 1 , \\ y'(t) &= 4x(t) - 7y(t) + 10t . \end{aligned}$$

The above procedure works, more generally, for any linear system of two equations and two unknowns with *constant coefficients* regardless of the order of the equations. For example, if we let L_1, L_2, L_3, and L_4 denote linear differential operators with constant coefficients (i.e., polynomials in D), then the method can be applied to the linear system

$$L_1[x] + L_2[y] = f_1 ,$$
$$L_3[x] + L_4[y] = f_2 .$$

Because the system has constant coefficients, the operators commute (e.g., $L_2L_4 = L_4L_2$) and we can eliminate variables in the usual algebraic fashion. Eliminating the variable y gives

$$(L_1L_4 - L_2L_3)[x] = g_1 , \quad (7)$$

where $g_1 := L_4[f_1] - L_2[f_2]$. Similarly, eliminating the variable x yields

$$(L_1L_4 - L_2L_3)[y] = g_2 , \quad (8)$$

where $g_2 := L_1[f_2] - L_3[f_1]$. Now if $L_1L_4 - L_2L_3$ is a differential operator of order n, then a general solution for (7) contains n arbitrary constants, and a general solution for (8) also contains n arbitrary constants. Thus, a total of $2n$ constants arise. However, as we saw in Example 3, there are only n of these that are independent for the system; the remaining constants can be expressed in terms of these.† The pair of general solutions to (7) and (8) written in terms of the n independent constants is called a **general solution for the system.**

If it turns out that

$$L_1L_4 - L_2L_3$$

is the zero operator, the system is said to be **degenerate.**

As with the anomalous problem of solving for the points of intersection of two parallel or coincident lines, a degenerate system may have no solutions, or if it does possess solutions, they may involve any number of arbitrary constants

Elimination Procedure for 2 × 2 Systems

To find a general solution for the system

$$L_1[x] + L_2[y] = f_1 ,$$
$$L_3[x] + L_4[y] = f_2 ,$$

where L_1, L_2, L_3, and L_4 are polynomials in $D = d/dt$:

(a) Make sure that the system is written in operator form.

(b) Eliminate one of the variables, say, y, and solve the resulting equation for $x(t)$. If the system is degenerate, stop! A separate analysis is required to determine whether or not there are solutions.

(c) (*Shortcut*) If possible, use the system to derive an equation that involves $y(t)$ but not its derivatives. [Otherwise, go to step (d).] Substitute the found expression for $x(t)$ into this equation to get a formula for $y(t)$. The expressions for $x(t)$, $y(t)$ give the desired general solution.

(d) Eliminate x from the system and solve for $y(t)$. [Solving for $y(t)$ gives more constants—in fact, twice as many as needed.]

(e) Remove the extra constants by substituting the expressions for $x(t)$ and $y(t)$ into one or both of the equations in the system. Write the expressions for $x(t)$ and $y(t)$ in terms of the remaining constants.